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On the neoclassical growth model with non-constant discounting

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Abstract

Krusell, Kuruşçu and Smith (Journal of Economic Theory, 2002) examined the neoclassical growth model with quasi-geometric discounting and demonstrated that the competitive economy performs better than the planning economy. However, recent experimental evidence supports other discounting functions such as hyperbolic or power discounting. In this note, we revisit their model without assuming specific functional forms for discounting and show that the competitive equilibrium always results in weakly higher welfare than the planning problem. Our conclusion is unchanged when we endogenize labor supply. The findings by Krusell et al. are therefore robust to the choice of discounting function.

Keywords: neoclassical growth; non-constant discounting; time-consistency

JEL classification: E5;

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1 Introduction

Experimental evidence (Thaler (1981), Benzion et al. (1989)) suggests that discounting of future rewards is not geometric. Economic models of time-inconsistent preferences are initially studied by Strotz (1956), Phelps and Pollak (1968) and Pollak (1968). These models are later reformulated by Laibson (1997) and Barro (1999) who adopt quasi-geometric (quasi-hyperbolic) discounting. Quasi-geometric discounting is also called $\beta - \delta$ preferences and the discounting function at time $t$ is given by $\beta \delta^t$, where $\delta$ is the discount factor and $\beta$ shows the degree of time inconsistency. Constant discounting corresponds to a case when $\beta = 1$.

An important study by Krusell, Kurusçu, and Smith (2002), KKS hereafter, introduces quasi-geometric discounting into the standard discrete-time neoclassical growth model. KKS solve the individual’s problem as the game between current self and future self, and focus on the Markov equilibria in which only the current state variables affect the individual’s behavior. Surprisingly, they show that the recursive competitive equilibrium path always welfare-dominates the social planner’s solution path.

Their results are, however, based on the assumption that discounting is quasi-geometric, and this assumption is inconsistent with certain recent economic experiments. $^1$ For example, a recent study Abdellaoui et al. (2010) uses a nonparametric method to measure the discounting function, and their evidence does not support for quasi-geometric discounting. Instead, they find that 40% of their subjects are consistent with geometric discounting and the remaining 60% are consistent with power discounting $(1 + t)^{-\phi}$ in Harvey (1986), which is a special case of hyperbolic discounting $(1 + \gamma t)^{-\phi}$ proposed by Loewenstein and Prelec (1992). $^2$

The objective of this note is to re-examine their results without assuming specific functional forms for discounting. As in KKS, we derive the recursive competitive equilibrium path and the planner’s solution path explicitly, by assuming the log period utility and

$^1$ Salois and Moss (2012) test quasi-geometric discounting $\beta \delta^t$ using market asset data, and find that the time inconsistency parameter $\beta$ is significantly different from 1. However, they assume that the discounting function is quasi-geometric and do not investigate other nongeometric discounting.

$^2$ If $\gamma = 1$, then hyperbolic discounting coincides with power discounting.
perfect capital depreciation. We then compare the competitive equilibrium path with the planner’s solution path in terms of social welfare and show that the competitive economy weakly welfare-dominates the planning economy for any discounting function. Our conclusion is unchanged when we consider labor-leisure choice of the agents. In this sense, the results in KKS are robust to the functional forms for discounting.

The quasi-geometric discounting function is now applied to many fields of economics, especially macroeconomics. Laibson (2001) argues that quasi-geometric discounting leads to undersavings, whereas Salanié and Treich (2006) demonstrate that the result can be the opposite for certain classes of utility functions. Diamond and Kőszegi (2003) show that workers with quasi-geometric discounting retire early. Karp (2002) examines an environmental problem. Schwarz and Sheshinski (2007) investigate social security issues. Indeterminacy results are obtained by Maliar and Maliar (2006). To the best of our knowledge, however, few studies investigate the neoclassical growth model with the general non-constant discounting function. Karp (2007) investigates a general class of discounting functions in the continuous time neoclassical growth models. However, he does not compare the planning economy with the competitive economy. Abdellaoui et al. (2010) argue that the extensive use of quasi-geometric discounting function in economics results form its analytical tractability, and not its empirical or experimental superiority over other discounting functions. Theoretical foundations of economic models with non-constant discounting are insufficient at this point.

The note proceeds as follows. Section 2 describes the model. Section 3 investigates the recursive competitive equilibrium path. Section 4 investigates the planner’s problem. Section 5 compares the equilibrium path with the planner’s solution path. Section 6 endogenizes the labor supply. Section 7 concludes. The Appendix contains proofs of the propositions.
2 The environment

The environment is the same as KKS except for the discounting function. Time is discrete and goes from 0 to $+\infty$. There is a continuum of individuals with unit measure. Preferences at the beginning of period 0 are given by

$$U_0 = \sum_{t=0}^{\infty} \lambda_t u_t = u_0 + \lambda_1 u_1 + \lambda_2 u_2 + \ldots,$$

where $u_t$ is the utility in period $t$ and $\lambda_t$ is the time weights in period $t$. In period 0, $\lambda_0 = 1$. Quasi-geometric discounting corresponds to a case with $\lambda_t = \beta \delta^t$, where $\delta$ denotes the instantaneous discount rate and $\beta$ denotes the degree of time-inconsistency. Abdellaoui et al. (2010) provide a parameter-free measurement of the discounting function. They find that the data does not provide a support for quasi-geometric discounting and that it is consistent with power discounting $\lambda_t = (1 + t)^{-\phi}$. Power discounting is originally proposed by Harvey (1986) and is a special case of hyperbolic discounting $(1 + \gamma t)^{-\phi}$ in Loewenstein and Prelec (1992). In the present note, the only restrictions on the time weights $\lambda_t$ are that they are non-negative and that the sum of $\lambda_t$, $\rho \equiv \sum_{t=1}^{\infty} \lambda_t$ is finite. Otherwise $U_0$ is not well-defined. Following KKS, we treat the individual's problem as the game between current self and future self, and focus on the Markov equilibria in which only the current state variables matter. The period utility from consumption $c$ takes the simple form $u(c) = \ln(c)$ and the intertemporal utility reduces to $U_0 = \sum_{t=0}^{\infty} \lambda_t \ln c_t$.

The production function is the Cobb-Douglas type. The labor supply of each agent is inelastic and is equal to one. Physical capital is fully depreciated and the resource constraint is

$$k_{t+1} = Ak_t^\alpha - c_t \text{ for } t \geq 0,$$

where $k_t$ is the stock of capital in period $t$, $A > 0$ represents the productivity and $\alpha \in (0, 1)$ denotes the capital share. The initial capital $k_0 > 0$ is given.

Factor markets are perfectly competitive and the real interest rate and the wage rate in period $t$ are given by $r_t = r(k_t) \equiv \alpha Ak_t^{\alpha-1}$ and $w_t = w(k_t) \equiv (1 - \alpha)Ak_t^\alpha$, respectively.
3 Recursive competitive equilibrium

In this section, we characterize the competitive economy. In the equations that follow, variables with primes show next-period values. The budget constraint of the individual is given by $k' = r(\bar{k})k + w(\bar{k}) - c$. He chooses his future state $k'$ by assuming that the factor prices $r(\bar{k})$ and $w(\bar{k})$ depend on the aggregate state $\bar{k}$, and that the process of the aggregate state $\bar{k}' = G(\bar{k})$ and his future decision rule $k' = g(k, \bar{k})$ are given. Here we seek an equilibrium in which $G$ and $g$ are time-invariant. The current self’s problem is

$$V_0^e(k, \bar{k}) = \max_{k'} \left[ \ln(r(\bar{k})k + w(\bar{k}) - k') + V^e(k', \bar{k}') \right],$$

where $V^e(k, \bar{k})$ is the value function after period 1. The current self believes that the future self will commit to adopt the decision rule $g$ after period 1. Thus $V^e$ is defined by

$$V^e(k, \bar{k}) = \sum_{t=1}^{\infty} \lambda_t \ln(c(k_t, \bar{k}_t)) = \lambda_1 \ln(c(k_1, \bar{k}_1)) + \lambda_2 \ln(c(k_2, \bar{k}_2)) + ..., \quad (3)$$

where $c(k, \bar{k}) = r(\bar{k})k + w(\bar{k}) - g(k, \bar{k})$ represents the consumption as a function of $k$ and $\bar{k}$.

If discounting is quasi-geometric and $\lambda_t = \beta \delta^t$, the value function $V^e$ coincides with the function $\beta \delta V(k, \bar{k}) = \beta \delta \{ \ln(c(k_1, \bar{k}_1)) + \delta \ln(c(k_2, \bar{k}_2)) + ... \}$ in KKS (p.51), but not $V$ itself. Both $V^e$ and $V$ represent the intertemporal utility after period 1. In the present study, the value is measured at the beginning of period 0 (see Eq. (2)) and then the utility in period 1 is discounted by $\lambda_1$. In contrast, in KKS, the value function $V$ is recursively defined as $V(k, \bar{k}) = \ln c(k, \bar{k}) + \delta V(g(k, \bar{k}), G(\bar{k}))$ and then the period-1 utility is not discounted. In fact, KKS define the problem of the current self at time 0, which corresponds to Eq. (2) as $\max_{k'} [\ln(r k + w - k') + \beta \delta V(k', \bar{k}')]$, and in this problem, $V$ is discounted by $\beta \delta$. Thus $V^e$ corresponds to $\delta \beta V$.

We closely follow KKS and define the recursive competitive equilibrium.

**Definition 1**: A recursive competitive equilibrium consists of an individual decision rule $g(k, \bar{k})$, a value function $V^e(k, \bar{k})$, the interest rate $r(\bar{k})$, the wage rate $w(\bar{k})$, and a
difference equation on the aggregate state $\bar{k}^{'} = G(\bar{k})$ such that

1. given $V^e(k, \bar{k})$ and $G(\bar{k})$, the rule $g(k, \bar{k})$ solves the optimization problem (2);
2. given $g(k, \bar{k})$ and $G(\bar{k})$, the function $V^e(k, \bar{k})$ satisfies Eq. (3);
3. $r(\bar{k}) = \alpha A\bar{k}^{\alpha-1}$ and $w(\bar{k}) = (1 - \alpha)A\bar{k}^{\alpha}$; and
4. $G(\bar{k}) = g(\bar{k}, \bar{k})$.

The fourth condition in Definition 1 means that the aggregate capital accumulation is consistent with the individual’s capital accumulation. Let $\theta = \sum_{t=1}^{\infty} \alpha^t \lambda_t$. As $\alpha < 1$, the parameters $\rho$ and $\theta$ satisfy $\rho > \theta$. We have the following proposition.

**Proposition 1** The recursive competitive equilibrium is given by:

1. $g(k, \bar{k}) = \frac{\rho}{\rho + 1}r(\bar{k})k$;
2. $G(\bar{k}) = g(k, \bar{k}) = s^eA\bar{k}^{\alpha}$, where $s^e = \frac{\alpha \rho}{\rho + 1}$ is the equilibrium savings rate; and
3. $V^e(k, \bar{k}) = (\theta - \rho)\ln \bar{k} + \rho \ln(k + \psi \bar{k}) + p^e$ where $\psi = (\rho + 1)(\frac{1}{\alpha} - 1)$ and $p^e$ is independent of $k$ and $\bar{k}$.

**Proof.** See the Appendix.

Proposition 1 implies that the savings rate is a time-independent constant. Let $\{k^e_t, c^e_t\}_{t=0}^{\infty}$ denote the equilibrium allocation where $k^e_t$ and $c^e_t$ denote physical capital and consumption in period $t$, respectively. It satisfies $k^e_{t+1} = s^eA(k^e_t)^{\alpha}$, and $c^e_t = (1 - s^e)A(k^e_t)^{\alpha}$. If discounting $\lambda_t = \beta \delta^t$ is quasi-geometric, $\rho = \sum_{t=1}^{\infty} \beta \delta^t = \frac{\beta \delta}{1 - \beta}$ and the savings rate becomes $s^e = \frac{\alpha \beta \delta}{1 - \beta + \beta \delta}$. This outcome concurs with KKS (see Proposition 1, p. 51).

4 Planning problem

In this section, we investigate the problem of the benevolent government. Following KKS, we assume that the planner’s preferences are the same as the ones of the current self, and we seek a time-consistent solution path to the planning problem. Assume that the current self perceives his future decision on physical capital by $g^*(k)$. The planner’s current self

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3It is also easy to verify that the function $V^e(k, \bar{k})$ coincides with $\delta \beta V(k, \bar{k})$ in KKS if $\lambda_t = \beta \delta^t$. 

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solves
\[ V_0^*(k) = \max_{k'} \left[ \ln(Ak^\alpha - k') + V^*(k') \right]. \] (4)
The value function after period 1, \( V^* \) satisfies
\[ V^*(k) = \sum_{t=1}^{\infty} \lambda_t \ln(c(k_t^*)), \] (5)
where \( c(k) = Ak^\alpha - g^*(k) \). A sequence of capital \( \{k_t^*\}_{t=0}^\infty \) is defined by \( k_{t+1}^* = g(k_t^*) \) for \( t \geq 1 \) and \( k_1^* = k \). Note that the function \( V^*(k) \) corresponds to the function \( \beta \delta V(k) \) in KKS (p.53). We formally define the planner’s solution.

**Definition 2:** A solution to the planner’s problem consists of a decision rule \( g^*(k) \) and a value function \( V^*(k) \), such that
1. given \( V^*(k) \), the rule \( g^*(k) \) solves the problem (4); and
2. given \( g^*(k) \), the value function \( V^*(k) \) satisfies Eq. (5).

We have

**Proposition 2** A solution to the planner’s problem is given by
1. \( g^*(k) = s^*Ak^\alpha \), where \( s^* = \frac{\theta}{\theta+1} \) is the savings rate; and
2. \( V^*(k) = \theta \ln k + p^* \) where \( p^* \) is constant.

**Proof.** See the Appendix. ■

If discounting \( \lambda_t = \beta \delta^t \) is quasi-geometric, \( \theta = \frac{\alpha \beta^t}{1-\alpha \beta} \). Thus the savings rate becomes \( s^* = \frac{\alpha \beta}{1-\alpha \beta(1-\beta)} \). This solution concurs with KKS (p. 53). Let \( \{k_t^*, c_t^*\}_{t=0}^\infty \) denote a solution path to the planner’s problem where \( k_t^* \) and \( c_t^* \) are respectively the period-\( t \) physical capital, and consumption. The solution satisfies \( k_{t+1}^* = s^*A(k_t^*)^\alpha \), and \( c_t^* = (1-s^*)A(k_t^*)^\alpha \).

Proposition 1 and 2 together imply that the competitive economy differs from the planning economy only in the savings rate. When discounting is geometric and \( \lambda_t = \delta^t \), \( s^e = s^* = \alpha \delta \) and the competitive equilibrium allocation is the same as the planner’s solution.
5 Utility comparison

Here we compare the equilibrium path with the planner’s solution path in terms of social welfare (i.e., utility of the current self). The following lemma characterizes the intertemporal utility of the individual whose savings rate is fixed.

Lemma 1 If the initial capital is \(k\) and the savings rate \(s\) is constant, the utility \(U_0 = \sum_{t=0}^{\infty} \lambda_t \ln c_t\) is separable in \(s\) and \(k\) and is given by

\[
U_0 = \alpha \frac{\rho - \theta}{1 - \alpha} \ln s + (\rho + 1) \ln(1 - s) + \alpha(\theta + 1) \ln k + \frac{\rho - \alpha \theta + 1 - \alpha}{1 - \alpha} \ln A. \tag{6}
\]

Proof. See the Appendix. ■

The intertemporal utility \(U_0\) corresponds to \(V_0^e\) in Eq. (2) and \(V_0^*\) (4). The best constant savings rate \(\hat{s}\) that maximizes \(U_0\) satisfies the first order condition \(\frac{\partial U_0}{\partial s} = 0\) and is given by

\[
\hat{s} = \frac{\alpha - \rho - \theta}{\rho - \alpha \theta + 1 - \alpha}.
\]

Note that, in general, the benevolent planner cannot choose \(\hat{s}\), because such a strategy is not time-consistent. We easily see that

\[
\alpha(\theta + 1)(s^* - \hat{s}) = \alpha \frac{(\rho + 1)\theta - \alpha \rho (\theta + 1)}{\rho - \alpha \theta + 1 - \alpha} = (\rho + 1)(s^e - \hat{s}).
\]

Therefore, the savings rates \(s^e\), \(s^*\) and \(\hat{s}\) satisfy

\[
s^* - \hat{s} = \Omega(s^e - \hat{s}), \tag{7}
\]

with \(\Omega = \frac{\rho + 1}{\alpha (\theta + 1)}\). Since \(\rho > \theta\) and \(\alpha < 1\), \(\Omega > 1\). Eq. (7) demonstrates that we have either \(\hat{s} \leq s^e \leq s^*\) or \(s^* \leq s^e \leq \hat{s}\) and the equality holds if and only if \(s^e = s^*\). Thus the equilibrium savings rate is closer to the welfare-maximizing savings rate than that chosen by the planner. It can easily be verified that \(s^* = s^e (= \hat{s})\) if and only if

\[
\alpha + \frac{1}{\sum_{t=1}^{\infty} \alpha^{t-1} \lambda_t} = 1 + \frac{1}{\sum_{t=1}^{\infty} \lambda_t}. \tag{8}
\]
Otherwise either \( \hat{s} < s^e < s^* \) or \( s^* < s^e < \hat{s} \) holds. When discounting is geometric and \( \lambda_t = \delta^t \), Eq. (8) is satisfied because both sides of Eq. (8) are equal to \( 1/\delta \). We have the following proposition.

**Proposition 3** The competitive economy performs weakly better the planning economy. The two economies coincide if and only if the discounting function \( \lambda_t \) satisfies Eq. (8).

KKS prove that when \( \lambda_t = \beta \delta^t \), Eq. (8) holds if and only if \( \beta = 1 \). It is not easy to specify the functional form for \( \lambda_t \) satisfying Eq. (8). We can at least note that certain non-constant discounting functions satisfy Eq. (8).

**Example 1** Assume that \( \lambda_1 = \sqrt{2\delta + (1 - \alpha)\delta^2 - (1 + \alpha)\delta} \), \( \lambda_2 = 2\delta \) and \( \lambda_t = 0 \) for \( t > 2 \). If \( \delta \) is sufficiently small, the discounting function \( \lambda_t \) satisfies \( 0 < \lambda_2 < \lambda_1 < 1 \) and Eq. (8).\(^4\) Therefore the competitive economy coincides with the planning economy.

When the discounting function is hyperbolic, we have the following proposition.

**Proposition 4** If the agent has hyperbolic discounting \( \lambda_t = (1 + \gamma t)^{-\phi} \) and the parameter \( \phi \) is sufficiently close to one, \( \hat{s} > s^e > s^* \) and the competitive economy performs strictly better than the planning economy.

**Proof.** See the Appendix. \( \blacksquare \)

If the agent has hyperbolic discounting, he is excessively impatient in the short run. The current planner recognizes that the curvature in the production function is concave and that returns to additional saving are diminishing. Thus the savings rate becomes excessively low. In the competitive equilibrium, current consumers are price-takers and the returns to saving are constant. Therefore the agents in the competitive equilibrium save more than the benevolent planner and the competitive economy does better than the planning economy.

\(^4\)If \( \lambda_t = 0 \) for \( t > 2 \), Eq. (8) holds if \( (\lambda_1 + \alpha \lambda_2)(\lambda_1 + \lambda_2) = \lambda_2 \).
6 Endogenous labor supply

So far the labor supply was inelastic. Here we endogenize the labor-leisure choice of the agents to check how our result changes. The utility is 
\[ \hat{U}_t = \sum_{t=0}^{\infty} \lambda_t \ln c_t + b \ln(1 - l_t), \]
where \( l_t \) is the labor supply in period \( t \) and \( b \ln(1 - l_t) \) with \( b > 0 \) shows the disutility from working. The resource constraint becomes \( k_{t+1} = A k_t \alpha l_t^{1-\alpha} - c_t \). The individual chooses his labor supply \( l \) and his future capital \( k' \), given the process of the aggregate capital \( \bar{k}' = H(\bar{k}) \) and that of labor \( \bar{l} = L(\bar{k}) \), the interest rate \( r(\bar{k}) = \alpha A \bar{k}^{\alpha-1} l^{-\alpha} \), the wage rate \( w(\bar{k}) = (1 - \alpha) A \bar{k}^{\alpha} l^{-\alpha} \), and his future decision rules on capital \( k' = h(k, \bar{k}) \) and on labor \( l = l(k, \bar{k}) \). The budget constraint is \( k' = r(\bar{k}) k + w(\bar{k}) l - c \).

A recursive competitive equilibrium consists of functions \( h, l, \hat{V}^e, r, w, H \) and \( L \) such that (1) given \( \hat{V}^e, H \) and \( L \), the decision rules \( h(k, \bar{k}) \) and \( l(k, \bar{k}) \) solves

\[
\hat{V}_0^e(k, \bar{k}) = \max_{k', \bar{l}, \bar{l}} \left[ \ln(r(\bar{k}) k + w(\bar{k}) l - k') + b \ln(1 - l) + \hat{V}^e(k', \bar{l}') \right],
\]

(2) given \( h, l, H \) and \( L, \hat{V}^e(k, \bar{k}) = \sum_{t=1}^{\infty} \lambda_t \ln(c(k_t, \bar{k}_t)) + b \ln(1 - l(k_t, \bar{k}_t)) \) with \( c(k, \bar{k}) = r(\bar{k}) k + w(\bar{k}) l(k, \bar{k}) - h(k, \bar{k}) \), (3) \( r(\bar{k}) = \alpha A \bar{k}^{\alpha-1} l^{-\alpha} \) and \( w(\bar{k}) = (1 - \alpha) A \bar{k}^{\alpha} l^{-\alpha} \) and (4) \( H(\bar{k}) = h(\bar{k}, \bar{k}) \) and \( L(\bar{k}) = l(\bar{k}, \bar{k}) \). We have the following proposition.

**Proposition 5** The recursive competitive equilibrium is given by (1) \( H(\bar{k}) = s^e A \bar{k}^{\alpha} (\bar{e})^{1-\alpha} \) and \( L(\bar{k}) = \bar{e} \) where \( s^e = \frac{\alpha \rho}{\rho + 1} \) and \( \bar{e} = \frac{\rho^e (\rho (\rho - 1) + b \rho + 1)}{\rho (\rho + 1) (1 - \alpha) (1 + b) + b \alpha} \), (2) \( h(k, \bar{k}) = \frac{\rho^e}{\rho + 1} r(\bar{k}) k \) and \( l(k, \bar{k}) = \bar{e} \{ 1 + \frac{b}{\eta - b} (1 - k/\bar{k}) \} \) with \( \eta = (\rho + 1)(\frac{1}{\alpha} - 1)(b + 1) + b \) and (3) \( \hat{V}^e(k, \bar{k}) = (\theta - \rho - b \rho) \ln \bar{k} + (1 + b) \rho \ln(k + \eta \bar{k}) + q^e \), where \( q^e \) is independent of \( k \) and \( \bar{k} \).

**Proof.** See the Appendix. \( \blacksquare \)

The planner’s problem consists of the decision rule on capital, \( k' = h^*(k) \), the one on labor \( l = l^*(k) \) and a value function \( \hat{V}^*(k) \), such that (1) given \( \hat{V}^* \), the rule \((h^*, l^*)\) solves

\[
\hat{V}_0^*(k) = \max_{k'} \left[ \ln(A k^\alpha l^{1-\alpha} - k') + b \ln(1 - l) + \hat{V}^*(k') \right],
\]

and (2) given \((h^*, l^*)\), \( \hat{V}^*(k) = \sum_{t=1}^{\infty} \lambda_t \ln(A(k^*_t)^\alpha(l^*_t)^{1-\alpha} - k^*_t) + b \ln(1 - l^*_t)) \), where \( k^*_t \)
and \( l^*_t \) are determined by \( k^*_{t+1} = h^*(k^*_t) \), \( k^*_1 = k \) and \( l^*_t = l^*(k^*_t) \). We have

**Proposition 6**  
A solution to the planner’s problem is given by (1) \( h^*(k) = s^* A k^\alpha (l^*)^{1-\alpha} \) and \( l^*(k) = l^* \) where \( s^* = \frac{\theta}{\sigma + 1} \) and \( l^* = \frac{(1-\alpha)(1+\theta)}{\beta + (1-\alpha)(1+\theta)} \) and (2) \( \dot{V}^*(k) = \theta \ln k + q^* \), where \( q^* \) is independent of \( k \).

**Proof.** See the appendix. □

When the savings rate \( s \) and labor supply \( l \) are constant, one has \( k_{t+1} = s(Al^{1-\alpha})k_t^\alpha \) and \( c_t = (1-s)(Al^{1-\alpha})k_t^\alpha \). The utility \( \hat{U}_0 = \sum_{t=0}^{\infty} \lambda_t (\ln c_t + b \ln (1-l)) \) reduces to

\[
\hat{U}_0 = \frac{\alpha(\rho - \theta)}{1-\alpha} \ln s + (\rho+1) \ln (1-s) + \alpha(\theta+1) \ln k + (\rho - \alpha \theta + 1 - \alpha) \ln l + b(\rho+1) \ln (1-l). \tag{9}
\]

(We ignore the constant term.) The best savings rate \( \hat{s} = \alpha \frac{\rho - \theta}{\rho - \alpha \theta + 1 - \alpha} \) is the same as before and the best labor supply is \( \hat{l} = \frac{\rho - \alpha \theta + 1 - \alpha}{b(\rho+1) + \rho - \alpha \theta + 1 - \alpha} \). Eq. (9) implies that \( \hat{U}_0 \) is determined by two variables: \( s \) and \( l \). One can easily show that with respect to each variable, the competitive equilibrium is closer to the first best than the planner’s solution. We have

**Proposition 7**  
The competitive economy always weakly welfare-dominates the planning economy even when the labor supply is endogenous.

**Proof.** See the Appendix. □

The result by KKS on the desirability of the competitive economy is therefore robust to the labor supply decision.

### 7 Concluding remarks

In this note, we study the neoclassical growth model with non-constant discounting, and demonstrate that the competitive equilibrium path always weakly welfare-dominates the planner’s solution path. This outcome implies that the original result in KKS is robust to a functional form for discounting. Future studies should specify the functional forms for discounting such that the competitive economy results in strictly higher welfare than the planning economy.
Appendix

The appendix provides proofs of the propositions. We let $r(\bar{k}) = \bar{r}$ and $w(\bar{k}) = \bar{w}$.

A Proof of Proposition 1

We adopt a guess-and-verify method. First, we show that given $V^e$ and $G$, the individual rule $g(k, \bar{k})$ solves Eq. (2), which reduces to

$$
\max_{k'} \ln(\bar{r}k + \bar{w} - k') + \rho \ln(k' + \psi k').
$$

The individual rule $k' = \frac{\rho}{\rho + 1} \bar{r}k$ satisfies the first order conditions $1/(\bar{r}k + \bar{w} - k') = \rho/(k' + \psi k')$ since the aggregate process is denoted as $\bar{k}' = \frac{\rho}{\rho + 1} \bar{r}\bar{k}$ and then

$$
\frac{1}{\bar{r}k + \bar{w} - k'} = \frac{1}{(\rho + 1)^{-1} \bar{r}k + \bar{w}} = \frac{\rho}{k' + \psi k'}.
$$

Note that $\bar{w} = \frac{1-\alpha}{\alpha} \bar{r}\bar{k}$.

Next we show that given $g$ and $G$, $V^e$ satisfies Eq. (3). To see this, first note that, $k_{t+1}/k_t = \bar{k}_{t+1}/\bar{k}_t = \rho/(\rho + 1)\bar{r}_t$. The consumption $c = (\rho + 1)^{-1} \bar{r}k + \bar{w}$ is denoted as $c = \bar{r}\bar{k}\{\rho + 1\}^{-1} \frac{k}{\bar{k}} + 1 - \frac{\alpha}{\alpha}$ and $k/\bar{k}$ is time-independent. Therefore

$$
\frac{c_{t+1}}{c_t} = \frac{\bar{r}_{t+1}k_{t+1}}{\bar{r}_tk_t} = \frac{\bar{k}_{t+1}}{\bar{k}_t} \frac{\bar{r}_{t+1}}{\bar{r}_t} = \frac{\rho}{\rho + 1} \bar{r}_{t+1},
$$

which means $\ln c_t = \ln c_1 + \sum_{i=2}^{t}(\ln \bar{r}_i + \ln \frac{\rho}{\rho + 1})$. Since $\ln \bar{r}_i = (\alpha - 1) \ln \bar{k}_i + \ln \alpha A$,

$$
\ln c_t = \ln c_1 - (1 - \alpha) \sum_{i=2}^{t} \ln \bar{k}_i + (t - 1) \ln (s^e A). \tag{10}
$$

The aggregate capital satisfies $\ln \bar{k}_i = \alpha \ln \bar{k}_{i-1} + \ln s^e A$ or equivalently $\ln \bar{k}_i = \alpha^{i-1} \ln \bar{k}_1 + \ln (s^e A)$. 

12
\[
\frac{1-\alpha}{1-\alpha} \ln s^e A. \text{ If } \bar{k}_1 = \bar{k} \text{ and } k_1 = k, \text{ we have}
\]
\[
(1 - \alpha) \sum_{i=2}^{t} \ln \bar{k}_i = (\alpha - \alpha^t) \ln \bar{k} + \left[ t - 1 - \frac{\alpha - \alpha^t}{1-\alpha} \right] \ln s^e A.
\]
Moreover, \( c_1 = (\rho + 1)^{-1} \bar{r}(k + \psi k) \) and then \( \ln c_1 = \ln \left( \frac{\alpha A}{\rho + 1} \right) + \ln (k + \psi k) + (\alpha - 1) \ln \bar{k} \).
Substituting these equalities into Eq. (10) yields
\[
\ln c_t = \ln \left( \frac{\alpha A}{\rho + 1} \right) + \ln (k + \psi k) - (1 - \alpha^t) \ln \bar{k} + \frac{\alpha - \alpha^t}{1-\alpha} \ln s^e A.
\]
Finally we get \( \sum_{t=1}^{\infty} \lambda_t \ln c_t = \rho \ln (k + \psi k) - (\rho - \theta) \ln \bar{k} + p^e, \) with \( p^e = \frac{\rho}{1-\alpha} \ln \left( \frac{\alpha A}{\rho + 1} \right) + \frac{\alpha - \theta}{1-\alpha} \ln s^e A, \) and it coincides with \( V^e(k, \bar{k}) \). Finally, the individual rule \( g \) is consistent with the aggregate process \( G \) since \( g(\bar{k}, k) = \rho / (\rho + 1) \bar{r} \bar{k} = G(\bar{k}). \)

**B Proof of Proposition 2**

We adopt the guess-and-verify method again. First, given \( V^* \), the planning problem is simplified as \( \max_{k'} \{ \ln (A\bar{k}^\alpha - k') + \theta \ln (k') \} \). The optimal value of \( k' \) coincides with \( g^* \).
Next, given \( g^* \), \( \ln k_{t+1} = \alpha \ln k_t + \ln s^e A \) and then \( \ln k_{t+1} = \alpha^t \ln k_1 + \frac{1-\alpha^t}{1-\alpha} \ln s^e A. \) Since \( c_t = (1 - s^*) \bar{k}^\alpha_t = \frac{1-s^*}{s^*} k_{t+1}, V^*(k) = \sum_{t=1}^{\infty} \lambda_t \ln c_t \) is written as
\[
V^*(k) = \theta \ln k + \rho \ln \left( \frac{1-s^*}{s^*} \right) + \frac{\rho - \theta}{1-\alpha} (\ln s^e + \ln A).
\]
Since \( s^* = 1 - (\theta + 1)^{-1} \), \( V^*(k) = \theta \ln k + p^* \) where \( p^* = \frac{\alpha - \theta}{1-\alpha} \ln \theta - \frac{\rho - \theta}{1-\alpha} \ln (\theta + 1) + \frac{\rho - \theta}{1-\alpha} \ln A \). Thus \( V^* \) satisfies Eq. (5).
C Proof of Lemma 1

If \( k' = sAk^\alpha \) and \( k_0 = k \), \( \ln k_{t+1} = \alpha^{t+1} \ln k + \frac{1-\alpha^{t+1}}{1-\alpha} \ln(sA) \). Thus \( c_t = (1-s)Ak^\alpha = (1/s - 1)k_{t+1} \) is expressed as

\[
\ln c_t = \ln(1-s) + \alpha^{t+1} \ln k + \alpha \frac{1-\alpha^t}{1-\alpha} \ln s + \frac{1-\alpha^{t+1}}{1-\alpha} \ln A.
\]

If we let \( \rho = \sum_{t=0}^{\infty} \lambda_t = 1 + \rho \) and \( \bar{\theta} = \sum_{t=0}^{\infty} \alpha^t \lambda_t = 1 + \theta \), \( U_0 = \sum_{t=0}^{\infty} \lambda_t \ln c_t \) is expressed as

\[
U_0 = \rho \ln(1-s) + \alpha \bar{\theta} \ln k + \alpha \frac{\rho - \bar{\theta}}{1-\alpha} \ln s + \frac{\rho - \alpha \bar{\theta}}{1-\alpha} \ln A.
\]

This implies Eq. (6). □

D Proof of Proposition 4

If discounting is hyperbolic, the parameter \( \theta \) satisfies

\[
\theta = \sum_{t=1}^{\infty} \frac{\alpha^t}{(1+\gamma t)^\phi} < \frac{\alpha}{(1+\gamma)^\phi} + \sum_{t=2}^{\infty} \frac{\alpha^t}{1+\gamma} < \frac{\alpha}{1+\gamma} + \frac{\alpha^2}{1-\alpha}.
\]

Thus \( \lim_{\phi \to 1} \theta \leq \frac{\alpha}{1+\gamma} + \frac{\alpha^2}{1-\alpha} < \frac{\alpha}{1-\alpha} \) and \( \lim_{\phi \to 1} s^* = \lim_{\phi \to 1} \frac{\theta}{1+\theta} < \alpha \). On the other hand, as \( \phi \) converges to 1, the sum of the discounting function \( \rho \) goes to \(+\infty\) and then \( \lim_{\phi \to 1} s^e = \lim_{\phi \to 1} \hat{s} = \alpha \). □

E Proof of Proposition 5

We first show that given \( \hat{V}^e \), the individual rules \( h \) and \( l \) solve the consumer’s problem, which reduces to

\[
\max_{l,k'} \{ \ln(rk + \bar{w}l - k') + b \ln(1 - l) + (1 + b)\rho \ln(k' + \eta \bar{k}') \}.
\]
The first order conditions are

\[
\left( \frac{1}{c} = \frac{1}{\bar{r}k + \bar{w}l - k'} \right) \frac{1}{b \bar{w}(1 - l)} = \frac{b}{\bar{w}(1 - l)} = \frac{\rho(1 + b)}{k' + \eta k'}. \tag{11}
\]

The production function is Cobb-Douglas and then \(\frac{\alpha \bar{w}^c}{(1-\alpha)k} = \bar{r} \). As \(\frac{1}{\eta} = \frac{\alpha c}{(\rho + 1)(1-\alpha)}\), one has \(\frac{\bar{w}}{\eta k} = \frac{\alpha \bar{w}^c}{(\rho + 1)(1-\alpha)k} = \frac{1}{\rho + 1} \bar{r} \) and \(l(k, \tilde{k}) = \frac{1}{1 + b} - \frac{\bar{w}k}{(\beta + 1)\eta k} \). Thus if \(k' = h(k, \tilde{k})\) and \(l = l(k, \tilde{k})\),

\[
\bar{w}l = \frac{1}{1 + b}(\bar{w} - \frac{b}{\rho + 1} \bar{r}k) \text{ and } \bar{r}k - k' = \frac{1}{\rho + 1} \bar{r}k.
\]

Therefore

\[
c = \frac{(\rho + 1)^{-1} \bar{r}k + \bar{w}}{b + 1} = \frac{\bar{r}(k + \eta \tilde{k})}{(b + 1)(\rho + 1)} = \frac{k' + \eta \tilde{k}'}{\rho(1 + b)}. \tag{12}
\]

and

\[
\bar{w}(1 - l) = \frac{(\rho + 1)^{-1} \bar{r}k + \bar{w}}{b + 1} = \frac{k' + \eta \tilde{k}'}{\rho(1 + b)}. \tag{13}
\]

Eqs. (12) and (13) together imply Eq. (11).

Next we show that given \(h, l, H\) and \(L, \hat{V}\) satisfies Eq. (3). Let \(\xi_j \ (j = 1, 2, ... 10)\) denote a constant which is independent of \(k\) and \(\tilde{k}\). Eq. (12) implies \(\frac{\xi_{t+1}}{c_t} = \frac{\rho}{\rho + 1} \bar{r}_{t+1}\) since \(k_t' = \frac{k_t'}{k} = \frac{\rho}{\rho + 1} \bar{r} \). Thus \(\ln c_t = \ln c_1 - (1 - \alpha) \sum_{i=2}^t \ln \tilde{k}_i + \xi_1 \). The aggregate capital evolves according to \(\ln \tilde{k}_i = \alpha \ln \tilde{k}_{i-1} + \xi_2 \). Thus we have \(\ln c_t = \ln c_1 - (\alpha - \alpha') \ln \tilde{k}_1 + \xi_3 \). From Eq. (12), consumption in period 1 is \(\ln c_1 = \ln (k_1 + \eta \tilde{k}_1) - (1 - \alpha) \ln \tilde{k}_1 + \xi_4 \). Therefore

\[
\ln c_t = \ln(k_1 + \eta \tilde{k}_1) - (1 - \alpha') \ln \tilde{k}_1 + \xi_5. \tag{14}
\]

Similarly, the wage rate evolves according to \(\bar{w}'/\bar{w} = \frac{\rho}{\rho + 1} \bar{r}'\) and then \(\ln \bar{w}_t = \ln \bar{w}_1 - (1 - \alpha) \sum_{i=2}^t \ln \tilde{k}_i + \xi_6 \) and

\[
\ln \bar{w}_t = \alpha' \ln \tilde{k}_1 + \xi_7. \tag{15}
\]

If \(k_1 = k\) and \(\tilde{k}_1 = \tilde{k}\), we use Eqs. (14) and (15) to obtain

\[
(1 + b) \ln c_t - b \ln \bar{w}_1 = (1 + b) \ln(k + \eta \tilde{k}) - (1 + b - \alpha') \ln \tilde{k} + \xi_8.
\]
From the first order condition (11), \( c_t = b^{-1} \bar{w}_t(1 - l_t) \). This implies

\[
\hat{V}(k, \bar{k}) = \sum_{t=1}^{\infty} \lambda_t (\ln c_t + b \ln(1 - l_t)) = \sum_{t=1}^{\infty} \lambda_t \{(1 + b) \ln c_t - b \ln \bar{w}_t\} + \xi_g,
\]

Thus \( \hat{V}^*(k, \bar{k}) = (1 + b)\rho \ln(k + \eta \bar{k}) - \{(1 + b)\rho - \theta\} \ln \bar{k} + \xi_{10} \) and this concurs Eq. (3). Finally, the individual rules are respectively consistent with the aggregate rules since \( h(k, \bar{k}) = \frac{\rho}{\rho + 1} \bar{r} \bar{k} = H(\bar{k}) \) and \( l(k, \bar{k}) = \bar{l} = L(\bar{k}) \).

**F Proof of Proposition 6**

First, given \( \hat{V}^* \), the planning problem reduces to \( \max_{k', l}\{\ln(Ak^\alpha l^{1-\alpha} - k') + b \ln(1 - l) + \theta \ln(k')\} \). One can easily check that the rules \( k' = h^*(k) \) and \( l = l^*(k) \) solve the problem. Next, given \( h^* \) and \( l^* \), \( \ln k_{t+1} = \alpha^t \ln k_1 + \frac{1-\alpha^t}{1-\alpha} \ln(s^* A l^* (1-\alpha)) \). Since \( c_t = \frac{1-s^*}{s^*} k_{t+1} \), \( \hat{V}^*(k_1) = \sum_{t=1}^{\infty} \lambda_t \{\ln c_t + b \ln(1 - l^*)\} \) is equal to \( \theta \ln k_1 \) plus constant.

**G Proof of Proposition 7**

If we let \( \Phi = (b + 1)(\rho + 1) - \alpha(\theta + 1) \), one has

\[
l^* - \hat{l} = b \frac{(\rho + 1)\theta - \alpha \rho(\theta + 1)}{\Phi\{b + (1 - \alpha)(\theta + 1)\}} = \frac{b}{1-\alpha} + \frac{(\rho + 1)^{1+b}}{\alpha} (l^* - \hat{l}).
\]

Since \( \rho \geq \theta \) and \( \frac{1+b}{\alpha} > 1 \), \( |l^* - \hat{l}| \geq |l^e - \hat{l}| \). Moreover, if \( (\rho + 1)\theta - \alpha \rho(\theta + 1) = 0 \), \( l^* = \hat{l} = l^e \) and \( s^* = \hat{s} = s^e \). Hence the savings rates \( s^* \), \( \hat{s} \) and \( s^e \) and labor supply \( l^* \), \( \hat{l} \) and \( l^e \) are either \( \hat{s} \leq s^e \leq \hat{s}^* \) and \( l \leq l^e \leq l^* \), or \( s^* \leq s^e \leq \hat{s} \) and \( l^* \leq l^e \leq \hat{l} \). In both cases, the competitive equilibrium is closer to the first best than the planner’s solution.
References


