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Makoto Yano and Yuichi Furukawa

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Makoto Yano and Yuichi Furukawa[2]
Kyoto University and Chukyo University

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[1]Corresponding author: Makoto Yano, Institute of Economic Research, Kyoto University, Yoshida-Honmachi, Sakyo-ku, Kyoto, 606-8051, Japan. Email: yano@kier.kyoto-u.ac.jp. Fax: 81-75-753-7198.

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Abstract

Many people argue that there have been three industrial revolutions over the last two hundred fifty years. At the same time, as Kondratieff shows, a period of deep stagnation have emerged in every fifty to sixty years, followed by that of active innovation. This study demonstrates that industrial revolution cycles and shorter Kondratieff-like long waves coexist along a single equilibrium path in an endogenous-exogenous growth model, in which an equilibrium system is ergodic chaos. This result explains industrial revolution cycles as a consequence of free entry, institutional intellectual property protection, and the interaction between endogenous and exogenous growth factors.

Keywords: industrial revolutions, chaotic cycles, intellectual properties, endogenous and exogenous growth.

JEL Classification Codes: C62; E32; O41

1 Introduction

It is often said that we are currently in the midst of the third industrial revolution. This observation leads to a question as to why a period of very fast and fundamental technological progress, which is referred to as an industrial revolution, have emerged cyclically just about every one hundred years. On the one hand, the first industrial revolution is often attributed to various institutional factors (North (1981, 1990) and Acemoglu, Johnson, and Robinson (2005)). On the other hand, however, the cause of industrial revolution cycles has scarcely been addressed in the existing literature.

Kondratieff (1925, 1935) discovers a similar phenomenon, i.e., the fact that a period of deep stagnation, followed by that of active innovation, emerges cyclically about every fifty to sixty years, i.e., in about half the length of an industrial revolution cycle. Rejecting stochastic elements as a cause of long waves, Kondratieff states,

> "In asserting the existence of long waves and in denying that they arise out of random causes, we are also of the opinion that the long waves arise out of causes which are inherent in the essence of the capitalistic economy" (see Kondratieff (1935, p. 115)).

This study demonstrates that industrial revolution cycles and shorter, and less drastic, Kondratieff-like long waves may emerge along a single equilibrium path. If we side with Kondratieff and look for a cause of industrial revolution cycles in the very "essence of the capitalistic economy," it is desirable to start with a model that suffers the least market frictions. For this reason, we search for causes in a dynamic general equilibrium model with an infinitely-lived representative consumer (Bewley (1982) and Yano (1984, 1998)).

This study reveals that industrial revolution cycles are attributable to the following three factors: Free entry, the patent system, and the interaction between exogenous knowledge accumulation and endogenous innovation. Following Matsuyama (1999, 2001), we assume that patents are protected only for one period. In order to capture the effectiveness of the patent system, we follow Helpman (1993) by assuming that only a certain percentage of new inventions can effectively be protected by patents. In order to explain the interaction between endogenous and exogenous growth factors, we incorporate Harrod-neutral technological progress into a standard endogenous growth model (Romer (1990), Grossman and Helpman (1991), and Aghion and Howitt (1992)).

Ergodic chaos theory is useful in order to explain industrial revolution cycles and Kondratieff-like long waves in a single dynamical system. As

Kondratieff (1925, 1935) emphasizes, a period of active innovation generally follows that of deep stagnation, resulting in long waves of innovation that are far more complicated than a regular periodic path. Ergodic chaos theory allows us to capture this process, in which a phase shift occurs repeatedly from a period of no innovation to that of active innovation. Occasionally, a period of "dead" stagnation is followed by a period of "explosive" innovation, achieving growth faster than the trend rate. We regard this as an industrial revolution, which repeatedly, but irregularly, emerges along an equilibrium path.

Chaotic industrial revolution cycles and Kondratieff-like long waves result from the fact that, in our model, an dynamical system consists of a downward-sloping part of active innovation and an upward-sloping part of no-innovation. Because non-expansiveness is an inherent feature of this system, the classical characterization of ergodic chaos for expansive systems is not applicable to our model.[1] We overcome this difficulty by a new result of Sato and Yano (2013), who show that an iteratively expansive unimodal system is ergodic chaos.

Matsuyama (2001) recognizes the possibility of innovation cycles in a frictionless market economy. He introduces an endogenous growth factor into the standard optimal growth model, based on an infinitely-lived representative consumer. He demonstrates the existence of period-2 cycles by assuming that inventions are protected for only one period; given that a patent protects an invention for twenty years, Matsuyama's period-2 innovation cycles is forty year long. Because of technical difficulties inherent in the optimal growth model, however, the existence of complex and chaotic long waves has not been known in the existing literature.

In our model, two opposing forces are at work behind chaotic industrial revolution cycles and Kondratieff-like long waves. The first force stems from Harrod-neutral exogenous technological progress, which increases the amount of effective labor force. This constantly enlarges the demand for new invention, because the larger the amount of effective labor force, the cheaper the production costs for manufactured goods. This in turn increases the demand for manufactured goods, thereby enlarging that for new inventions. The second force stems from technological progress through endogenous innovation. This expands the number of manufacturing sectors producing differentiated products. By the smoothing effect of the intertemporal optimization of a consumer, the demand for consumption goods tends to be spread out over time. This implies that the larger the number of products at the beginning of

[1] That follows from Lasota and Yorke (1973), Kowalski (1975), and Li and Yorke (1978); see Bhattacharya and Majumdar (2007) and Grandmont (2008) for details.

a period, the smaller the demand for differentiated products, which reduces the willingness to pay for a new invention in that period. On the one hand, in short, variety increasing endogenous technological progress reduces the willingness to pay for an invention. On the other hand, labor-economizing exogenous technological progress enlarges it. These two opposing forces result in chaotic industrial revolution cycles, the average length of which is three to four periods; this implies that each wave has a length of about sixty to eighty years, for patents are usually protected for about twenty years.

This mechanism differs from creative destruction, proposed by Schumpeter (1942) to explain Kondratieff-like long waves. In the modern literature, Francois and Lloyd-Ellis (2003) capture this Schumpeterian effect by using in the quality-ladder model of Grossman and Helpman (1991). Although its focus is more heavily on industrial revolution cycles, our study complements Francois and Lloyd-Ellis (2003), as well as Matsuyama (2001), in providing alternative explanations for innovation cycles in a broad sense.

It is not our intention to jump in the long debate as to whether the first industrial revolution was brought about by an exogenous cause or endogenous. Our intention is, rather, to show that, after the first industrial revolution occurred for one reason or another, the subsequent instability in innovation dynamics, which are exhibited in industrial revolution cycles and Kondratieff-like long waves, may be attributable to various institutional factors.

Because the assumption of an infinitely-lived consumer keeps a model free from a friction, an optimal growth model has been adopted in order to study complex economic dynamics (Benhabib and Nishimura (1979, 1985)).[2] Despite this, only a few specific models have been known in which optimal paths follow chaotic dynamics; see Boldrin and Deneckere (1990), Nishimura and Yano (1995a, 1996a) and Khan and Mitra (2005, 2012).[3] For the same reason, many studies have dealt with the reverse problem, which seeks for a dynamic model in which equilibrium paths follow a pre-determined chaotic dynamical system; see Boldrin and Montrucchio (1986), Deneckere and Pelikan (1986), Nishimura, Sorger, and Yano (1994), Nishimura and Yano (1996b), Mitra (1996), and Mitra and Sorger (1999). In capturing chaotic economic dynamics that results from the existence of a lower bound for a state variable that is flatter than 1, this study extends the line of studies by Nishimura and Yano (1995b) and Baierl, Nishimura and Yano (1998) and Sato and Yano

[2]For an early application of chaos theory to cyclical capital accumulation outside of optimal growth, see Benhabib and Day (1980) and Grandmont (1985).

[3]Few examples of chaotic growth with an infinitely-lived consumers have been known even outside of optimal growth theory; for exceptions, see Matsuyama (1991), Fukuda (1993), and Boldrin, Nishimura, Shigoka, and Yano (2001).

(2013), who show slow capital depreciation and Khan and Mitra (2012) for what they call the Robinson-Solow-Srinivasan model.

Outside of the literature on infinitely-lived consumer, it has been demonstrated by many studies that the number of new inventions fluctuate over time and, sometimes, chaotically. See Shleifer (1986), Gale (1996), and Matsuyama (1999) for periodic innovation cycles and Deneckere and Judd (1992), Mitra (2001), Mukherji (2005), and Yano, Sato, and Furukawa (2011)) for chaotic innovation cycles.

In Section 2 below, first, we will explain the use of ergodic chaos theory in capturing industrial revolution cycles and Kondratieff's long waves. In Section 3, we will build our dynamic model. In Section 4, we will explain the two deriving forces of innovation cycles. Section 5 is concerned with the existence of Kondratieff-like long waves. In Section 6, conditions for chaotic industrial revolution cycles are obtained in a limit case. Section 7, we will show that the patent system is a necessary condition for innovation cycles in our model.

2 Chaotic Industrial Revolution Cycles

The main contribution of this study is to explain industrial revolution cycles and their coexistence with less drastic Kondratieff-like long waves along a single equilibrium path in a macroeconomic model that is compatible with the dynamic optimization of an infinitely-lived consumer. For this purpose, the theory of ergodic chaos is highly useful. At the outset, we explain this fact along with the basic working of our model.

Ergodic chaos theory provides a standard tool for explaining irregular cycles along a dynamic path. Kondratieff (1925) observed fifty to sixty year waves of innovation. Ergodic chaos theory makes it possible to explain big bursts of new innovation, which are commonly referred to as industrial revolutions, in the same context as Kondratieff's long waves.

In order to explain this phenomenon by ergodic chaos, we set our model in discrete time and assume that the length of an individual period is equal to that in which patents are effective. This implies that the length of a period is about twenty years, if new inventions do not become obsolete before their patents expire. If many become obsolete before their patents expire, the length of an individual period may be thought of as about ten to twenty years.

2.1 Endogenous and exogenous growth factors

We build an endogenous-exogenous growth model, in which labor productivity increases both endogenously and exogenously. That is, in our model, labor productivity increases due to innovation expanding the number of technologies for differentiated products. At the same time, labor productivity increases due to Harrod-neutral exogenous technological progress.

Let α be the exogenous technological progress rate, i.e., that the effective labor in period t is

$$E_t = \alpha^t \bar{E} > 0. \tag{1}$$

With this assumption, we think of the case in which not physical labor but the quality of labor increases at the constant rate.

Assume that an invention made in a period becomes available for production in the subsequent period. Denote as Z_t the number of differentiate products that are newly invented in period t and will become available for production in period $t+1$. Denote as N_t that of cumulative invented technologies from the past through period $t-1$. That is, Z_t and N_t obey the following dynamics:

$$N_t = N_{t-1} + Z_{t-1}. \tag{2}$$

Assume that at the beginning of period 1 (time 0), a positive number of differentiated products invented before the initial period $N_0 = \overline{N} > 0$ exists. Denote as $Z_0 = \overline{Z} \geq 0$ the number of newly invented differentiated product at time 0, which can be zero. Technological progress is irreversible with respect to both types of technologies. In the case of endogenous technological progress, therefore, it must hold

$$Z_{t-1} \geq 0. \tag{3}$$

In the system captured by (1) - (3), growth is driven both by an endogenous factor (represented by $Z_{t-1} > 0$) and an exogenous factor (represented by $E_t = \alpha^t \bar{E}$, $\alpha > 1$).

2.2 Capturing complex nonlinear dynamics

Equations (1) through (3) summarize the basic dynamic framework of our model, involving three state variables, N_t, Z_t and E_t. In order to complete the dynamical system, we need to specify the process in which Z_t evolves. The dynamics of Z_t is specified by our macroeconomic model that is based on the optimization of an infinitely-lived representative consumer and of firms in the retail sector, the manufacturing sector, and the R&D sector. This dynamics can be expressed by

$$Z_t = Z(N_{t-1}, Z_{t-1}, E_{t-1}).^4 \tag{4}$$

Equations (1), (2), and (4) constitutes an non-autonomous dynamical system of N_t, Z_t and E_t.

Once we build the dynamic model in Section 3, we will transform it into an autonomous model; since equations (1) through (3) are non-autonomous, so is the macroeconomic model, built on those equations. This transformation is done by normalizing state variables N_t and Z_t by L_t, i.e., focusing on $n_t = N_t/(\alpha^t L)$ and $z_t = Z_t/(\alpha^t L)$. These normalized variables represent the "effective numbers" of cumulative and newly invented technologies, respectively, N_t and Z_t. With these variables, we may transform (1) and (2) as follows:

$$n_t = \frac{1}{\alpha}(n_{t-1} + z_{t-1}). \tag{5}$$

Note that (3) implies

$$z_t \geq 0. \tag{6}$$

As in the standard dynamic general equilibrium model in which consumers are assumed to be infinitely-lived (Yano, 1998), an equilibrium path is characterized by a set of Euler equations (Euler system). The Euler system, derived from first order conditions of optimization for consumers and firms, may be expressed as a recursive system of the state variables and their co-state variables (prices of state variables). In our model, it may be written as

$$\begin{pmatrix} n_t \\ z_t \\ q_{nt} \\ q_{zt} \end{pmatrix} = \begin{pmatrix} E_n^o(n_{t-1}, z_{t-1}, q_{nt-1}, q_{zt-1}) \\ E_z^o(n_{t-1}, z_{t-1}, q_{nt-1}, q_{zt-1}) \\ E_{q_n}^o(n_{t-1}, z_{t-1}, q_{nt-1}, q_{zt-1}) \\ E_{q_z}^o(n_{t-1}, z_{t-1}, q_{nt-1}, q_{zt-1}) \end{pmatrix}, \tag{7}$$

where q_{nt} and q_{zt} are the co-state variables associated with n_t and z_t, respectively. An equilibrium path must satisfy this system, (7), as well as the initial conditions of the economy or, in other words, $n_0 = \bar{n} = \bar{N}/\bar{E}$ and $z_0 = \bar{z} = \bar{Z}/\bar{E}$. The Euler system cannot, however, determine an equilibrium path, since a specific solution to (7) cannot be determined unless the initial values for the co-state variables, q_n and q_z, are set. The initial values for the co-state variables are determined endogenously in the underlying general equilibrium model by the transversality condition. If the initial conditions for co-state variables can be solved explicitly, it is possible to obtain a specific

[4] This expression can be derived from (33).

dynamical system that characterizes the equilibrium dynamics of a model, which may be expressed, in our case, as

$$\begin{pmatrix} n_t \\ z_t \end{pmatrix} = \begin{pmatrix} \frac{1}{\alpha}(n_{t-1} + z_{t-1}) \\ z(n_{t-1}, z_{t-1}) \end{pmatrix}. \tag{8}$$

In the existing literature, there are two approaches to capture business cycles in a macroeconomic model with infinitely-lived consumers. The first is to search for periodic solutions to the Euler system. A periodical equilibrium path is obtained by solving (7) for a stationary state, in which $(n_t, z_t, q_{nt}, q_{zt}) = (n_{t-1}, z_{t-1}, q_{nt-1}, q_{zt-1})$. Since Benhabib and Nishimura (1979), their method has been widely adopted in the literature on business cycles (e.g., Matsuyama (2001)). Powerful as it is, however, this does not prove the possibility of a complex equilibrium path from a given initial condition, (n_0, z_0); rather, it merely proves that if the initial condition happens to be at a periodic point, the equilibrium path from that initial condition will behave cyclically.

In order to demonstrate the existence of business cycles for arbitrary initial conditions, equilibrium dynamics must be characterized globally rather than on periodic paths. Once that is done, it is possible to demonstrate the possibility of irregular (chaotic) business cycles; see Boldrin and Montrucchio (1986) and Deneckere and Pelikan (1986)). Extending their study, Nishimura and Yano (1994, 1995a) develop a method of explicitly deriving a specific equilibrium dynamical system by using the feasibility condition of a model. We take an approach similar to Nishimura and Yano (1995a).

2.3 Industrial revolution cycles and Kondratieff-like long waves in chaotic dynamics

Two merits exist in proving that the equilibrium dynamical system is ergodic chaos. First, we may demonstrate the existence of permanent fluctuations along an equilibrium path for almost every initial condition. Second, and more importantly, ergodic chaos theory permits to capture many different patterns with respect to phase changes between a period of no innovation and that of active innovation. A period of no innovation may be followed sometimes by a period of very rapid innovation, in which the growth rate exceeds the trend rate by far, and some other times by a period of slow innovation in which the growth rate stays below the trend rate. Yet in other cases, the level of innovation may simply fluctuate without falling in a no-innovation period. In our model, these phenomena emerge along a single equilibrium path. This explains the real world history that industrial revolution cycles and less drastic, Kondratieff-like long innovation waves have alternated with highly irregular patterns over the last two hundred fifty years.

One contribution of the present study is to render this possible by building a macroeconomic model with infinitely-lived consumers; it is infeasible to show the existence of ergodic chaos in a given economic model without properly specifying explicit functional forms for utility and production functions.

An equilibrium dynamical system is a recursive system with two state variables, n_t and z_t, the chaotic behavior of which is analytically intractable. However, we build a model in such a way that if the value of a particular parameter is taken to a limit, the system will be boiled down to a dynamical system with a single state variable; see $n_t = \eta(n_{t-1})$ given by (43), and (44). This limit system renders tractable chaotic industrial revolution cycles in Kondratieff-like long waves.

In Figure 1, a typical graph of the equilibrium dynamical system, η, is illustrated by the kinked line $X_L X^o X_H$. Let b^o be the point at the kink, X^o. This point determines the practical lower and upper bounds of the system, y_L and y_H. Although it is necessary to demonstrate that the system is in fact ergodic chaos, it is easy to see that any path generated by this system follows highly complicated nonlinear dynamics.

In our model, the growth rate of invented technologies is equal to

$$g_t = N_{t+1}/N_t = \alpha n_{t+1}/n_t. \tag{9}$$

In Figure 1, half segment $X^o X_H$ lies line $\Lambda : n_t = \frac{1}{\alpha} n_{t-1}$. If (n_t, n_{t+1}) is on $X^o X_H$, $z_t = 0$ by (8), which implies that no inventions are made in period t and that the growth rate is $g_t = 1$. The steady state of the system is at the intersection between half segment $X_L X^o$ and the 45 degree line, at which $n_t = n_{t-1}$. Thus, by (9), the state number of technologies N_t grows at the rate of α in the steady state.

Now, think of an equilibrium path starting from $n_0 = b^o$ in Figure 1. Along this equilibrium path, no innovation takes place in period 1 ($g_1 = \alpha y_L / b^o = 1$), and the highest growth rate will be achieved in period 2 ($g_2 = \alpha y_H / y_L$). In period 3, the no innovation phase will emerge again, in which the economic activities are described by $(y_H, \eta(y_H))$. In period 4, although new inventions are created, a growth rate will remain below the steady state rate, α. After that period, the effective number of new inventions, n_t, slowly fluctuate without experiencing a period of no innovation until period 8, in which innovation will cease again. In period 9, however, a growth rate higher than the steady state rate will be achieved.

As is discussed in the Introduction, Kondratieff (1925, 1935) observes that a period of active innovation follows that of deep stagnation. If we are to treat industrial revolution cycles in the same context, it is natural to regard as an industrial revolution a burst of new innovation following a period of stagnation. Although in a highly stylized fashion, Figure 1 describes such a phenomenon by a phase shift from a period of no innovation to that in which new inventions are created at a rate faster than the steady state growth rate, α. Along the equilibrium path from b^o, an industrial revolution in this sense occurs in periods 2 and 9. From period 3 to period 8, the number of new innovations fluctuates slowly, thereby generating less drastic Kondratieff-like long waves. In Figure 1, those industrial revolution phases are indicated by solid arrows, and Kondratieff-like slow and long waves by dotted arrows.

2.4 Mathematical Preparation

Our limit equilibrium is not expansive, i.e., has a slope smaller than 1. For this reason, the standard characterization for ergodic chaos (the Lasota-Yorke-Kowalsk-Li theorem) cannot be employed in our system. In order to characterize ergodic chaos in a non-expansive system, therefore, we adopt a recent result by Sato and Yano (2013), which implies that an iteratively expansive unimodal system is strong ergodic chaos.

For the sake of explanation, let I be a closed interval in $\mathbb{R}$. Adopt the convention $f^t = f \circ f^{t-1}$, $t = 1, 2, \ldots$, $f^0(x) = x$ and $f^1(x) = f(x)$. A dynamical system $f : I \to I$ is said to be expansive if it is piecewise twice continuously differentiable and if $\inf |f'(x)| > 1$ over the set of x at which $f'(x)$ is well defined. Moreover, f is iteratively expansive if there is n such that $f^n(x)$ is expansive (Lasota and Yorke, 1973). Finally, a system, $f : I \to I$ is unimodal if it is continuous and if there is $c \in I$ either such that $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$ or such that $f'(x) < 0$ for $x < c$ and $f'(x) > 0$ for $x > c$, whenever f' is defined.

Denote as m the Lebesgue measure. A measure μ on I is absolutely continuous (with respect to the Lebesgue measure) if $m(A) = 0$ implies $\mu(A) = 0$. Moreover, a system, f, is said to be invariant if for any μ-measurable $A \subset I$, $f^{-1}(A) = A$ implies $\mu(A) = \mu(f^{-1}(A))$. It is ergodic with respect to a measure μ (or μ-ergodic) if $\mu(A) = \mu(f^{-1}(A))$ implies $\mu(A) = 0, 1$.

In the standard literature (Grandmont (1986, 2008) and Bhattacharya and Majumdar (2007)), ergodic chaos is defined as a system $f : I \to I$ that can be associated with a Lebesgue-absolutely continuous, invariant probability measure with respect to which the system is ergodic, μ. In contrast, Sato and Yano (2013) introduce strong ergodic chaos, which is defined as a system $f : I \to I$ that can be associated with a Lebesgue-absolutely continuous, invariant probability measure and is ergodic with respect to the Lebesgue measure, m.

Ergodic chaos theory regards an associated invariant measure, μ, as an endogenous object determined by the fundamental structure of system f, and seeks for conditions under which a system is ergodic chaos. Lasota and Yorke (1973), Kowalski (1975), and Li and Yorke (1978) provide a strong and handy characterization for ergodic chaos, which hinges on the expansibility of a system. Their result implies that a system is ergodic chaos if it is expansive, unimodal and continuously twice differentiable except at the peak of the map, c.

In their recent study, Sato and Yano (2013) provide a new characterization of ergodic chaos, which permits a system to be non-expansive. Their result implies the following:

Proposition 1 (Sato-Yano) *If a system $f : I \to I$ is iteratively expansive and unimodal, it is strong ergodic chaos. Moreover, an invariant measure associated with the system, f, is uniquely determined.*

Ergodic chaos implies that solutions to a dynamical system behave as if they were generated by a random process. Strong ergodic chaos has a stronger implication than standard ergodic chaos in that the region in which the so-called Birkhoff-von-Neumann identity holds is larger (or, more precisely, the domain of the system itself, I, rather than the support of the invariant measure associated with the system, support μ).

Birkhoff (1931) and von Neumann (1932) show the existence of a natural coupling between a deterministic system, $f : I \to I$, and a probability measure on I, μ, such that the action average can be related to the space average, i.e., for any measurable $A \subset I$, it holds that

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \chi_A(f^t(x_0)) = \mu(A), \qquad (10)$$

where χ is the characteristic function defined by $\chi_A(x) = 1$ if $x \in A$ and $= 0$ if $x \notin A$. In particular, Birkhoff (1931) proves that if the system is ergodic with respect to the invariant measure, μ, equation (10) holds for almost every A with respect to the invariant measure, μ.

Sato and Yano (2013) demonstrate a stronger result, showing the ergodic identity holds for Lebesgue-almost every initial state. That is,

Proposition 2 *In strong ergodic chaos, the ergodic identity, (10), holds for Lebesgue-almost every initial state between the system, f, and the associated invariant measure.*

3 Dynamic Model

Assume that, in each period, each R&D firm decides whether or not to invest in inventing a new technology for producing a differentiated middle product. Inventions are produced by using only labor. Let κ be the labor input needed for making a new invention. By using this technology, each manufacturer produce a differentiated middle output; let λ be the labor input needed to make one unit of a middle product. The retail sector transforms the middle products into a single final consumption good.

In order to examine the role of intellectual property protection in industrial revolution cycles, we assume that in each period, only 100ϕ percent of the new inventions are actually protected. This assumption follows Helpman (1993) and is adopted to demonstrate in the simplest fashion that a sufficient level of intellectual property protection is a necessary condition for industrial revolution cycles. As is discussed in the Introduction, parameter ϕ may be interpreted as capturing the enforcement level of the patent rule. It can be interpreted also as the standard that the patent authority applies to patent applications.[5] Under this assumption, the numbers of monopolistic and competitive markets in period t are, respectively, $N_t^M = \phi Z_{t-1}$ and $N_t^C = (1-\phi)Z_{t-1} + N_{t-1}$.

There are four types of firms in the economy. They are perfectly competitive retail firms, perfectly competitive manufacturers, monopolistic manufacturers, and innovation firms. Consumers, represented by a single agent, consume the final consumption good, sold by the retail sector. The representative consumer chooses a sequence of consumption, X_t, so as to maximize the following intertemporal utility,

$$U = \sum_{t=1}^{\infty} \beta^{t-1} \ln X_t, \tag{11}$$

where $0 < \beta < 1$. By normalizing the current value of consumption goods in each period to 1 and denoting as r_j the real interest rate in period t, the consumer's wealth constraint is expressed as

$$\sum_{t=1}^{\infty} \left(\prod_{j=1}^{t-1} \frac{1}{1+r_j} \right) X_t \leq W_0 \tag{12}$$

with the wealth, W_0, consisting of the sum of discounted values of labor and

[5] Under this interpretation, it is implicitly assumed that patentabilities differ across inventions, although the differentiated products, produced from invented technologies, are assumed to be symmetric.

his initial asset or, in other words, equal to

$$W_0 = \sum_{t=1}^{\infty} \left(\prod_{j=1}^{t-1} \frac{1}{1+r_j} \right) w_t \alpha^t \bar{E} + A_0, \tag{13}$$

where A_0 is the value of initial asset at the beginning of period 1.

The retail sector, in period t, transforms the existing differentiated products (i.e., those in closed interval $[0, N_t]$) into the final consumption good in that period. Denote as X_t the amount of the final consumption good produced in period t and as x_{tj} the amount of good j employed to produce X_t. For the retail sector's production function, we adopt the standard CES function,

$$X_t = \left(\int_0^{N_t} {x_{tj}}^{1-\theta} dj \right)^{\frac{1}{1-\theta}}, \quad 0 < \theta < 1; \tag{14}$$

see Dixit and Stiglitz (1977) and Ethier (1982). Note that θ is the inverse of the elasticity of substitution between any two differentiated products. The retail sector's optimization problem can be expressed as, for each period,

$$\max_{x_{tj}} \left[\left(\int_0^{N_t} {x_{tj}}^{1-\theta} dj \right)^{\frac{1}{1-\theta}} - \int_0^{N_t} p_{tj} {x_{tj}}^{1-\theta} dj \right], \tag{15}$$

where p_{tj} is the price of product j in period t. This implies that the input demand for differentiated product j in period t can be expressed by the following inverse demand function

$$p(x_{tj}; X_t) = (X_t / x_{tj})^{\theta}. \tag{16}$$

Next, we will describe the market for patent licenses. In each period, an innovation firm can invent one technology to produce a new differentiated product by using κ units of labor. Denote as (t, i) the ith invention that is invented in period t. This invention can be utilized to produce a product in period $t+1$, which we call (t, i) as well. The innovation firm to invent technology (t, i) can sell the licence for its invention at price P_t if the invention is not to be free ridden and at 0 if it is to be free ridden. Denote as $\tilde{P}_t$ the distribution of this price.

Each innovation firm decides whether or not to employ workers for an invention. If, in period t, an innovation firm makes input for (t, i), denote its choice as $\delta^I_{(t,i)} = 1$. If it does not, denote the choice as $\delta^I_{(t,i)} = 0$. Thus, the profit maximization problem of the innovation firm to invent technology $(t-1, i)$ can be written as

$$\max_{\delta^I_{(t-1,i)} \in \{0,1\}} \mathcal{E} \left(\frac{\tilde{P}_t}{1 + r_{t-1}} - w_t \kappa \right) \delta^I_{(t-1,i)}, \tag{17}$$

where $\mathcal{E}$ is the operator taking the expected value of a random variable.

In each period, the manufacturing firm that purchases the patent licence for a newly invented technology can sell the product monopolistically in that period. Each individual manufacturer in the market for new inventions decides whether or not to invest in a licence for an invention ($\delta_{tj}^M = 1$ or 0) and how many units of the product it will produce by using the invention (x_{tj}^M), where $j = (t-1, i)$. Assume that, in doing so, it takes the manufacturing sector's aggregate output, X_t, as given. The manufacturer must pay P_{tj} for the license in purchasing the license of invention $j = (t-1, i)$. In addition, it must pay $w_t \lambda x_{tj}^M$ for labor to produce output since, by assumption, λ units of labor per unit of output are needed. Thus, the optimization problem of a manufacturer, acquiring $j = (t-1, i)$ invention, is

$$\max_{\delta_{tj}^M \in \{0,1\}, x_{tj}^M \geq 0} (p(x_{tj}^M; X_t)x_{tj}^M - w_t \lambda x_{tj}^M - P_{tj})\delta_{tj}^M. \quad (18)$$

In an equilibrium in the market for licenses for patent, each new invention that is not free ridden must be purchased by a manufacturer. This implies that in an equilibrium in the period-t market for licenses of inventions, it holds that

$$\delta_{tj}^M = \delta_{(t-1,i)}^I \text{ if invention } j = (t-1, i) \text{ is not free ridden.} \quad (19)$$

We assume that free entry is guaranteed on both sides of the market for intellectual properties. On the demand side, this implies that the profit each monopolistic manufacturer can acquire by using the exclusive license for a technology must not exceed the price of a license. By (18), this can be written as

$$(p(x_{tj}^M; X_t) - w_t \lambda)x_{tj}^M \leq P_{tj}, \quad (20)$$

where strict equality holds if $\delta_{tj}^M > 0$. On the supply side, the expected present value of a price of a licence must not exceed the opportunity cost of an invention for an innovation firm. Since this opportunity cost is $w_{t-1}\kappa$, this condition can be expressed as

$$\frac{\phi P_{tj}}{1 + r_{t-1}} \leq w_{t-1}\kappa, \quad (21)$$

where strict equality holds if $\delta_{(t-1,i)}^I > 0$. In summary, the market for intellectual properties is described by (17)-(21).

The competitive manufacturing sector in period t consists of firms that use the technologies invented before period $t-2$ and those that are invented in period $t-1$ but failed to receive the patent protection. The optimization problem in a perfectly competitive manufacturing sector can be written as

$$\max_{x_{tj}^C} (p_{tj}^C x_{tj}^C - w_t \lambda x_{tj}^C), \tag{22}$$

where x_{tj}^C and p_{tj}^C, respectively, denote the amount and price of output produced by the competitive sector using publicly known technology j.

In the labor market, labor is employed by the manufacturing sector (monopolistic and competitive) and the R&D sector. Thus, the market clearing condition is

$$\int_0^{N_t} \lambda x_{tj} dj + \kappa Z_t = \alpha^t \bar{E}, \tag{23}$$

which implies that the sum of the amount of effective labor forces employed by the manufacturing sector and that employed in the invention sector must be equal to the existing amount of effective labor forces. This completes the description of our model.

4 Driving Forces of Innovation Cycles

The model above has two state variables, Z_t and N_t. One important feature of the model is that, even though it is based on the intertemporal optimization of an infinitely-lived consumer, unlike an optimal growth model, the equilibrium system can be written as a dynamical system of the two state variables independently of the co-state variables (prices); that system automatically satisfies the transversality condition for intertemporal optimization.

This dynamical system captures the two driving forces of cyclical innovation dynamics: (i) Harrod-neutral exogenous growth, which constantly pushes up the demand for inventions, and (ii) endogenous inventions, which create new manufacturing industries, spread out the demand for inventions as a whole, thereby reducing the demand for each individual invention.

Due to symmetricity, $x_{tj}^M = x_t^M$ and $p_{tj}^M = p_t^M$ for all monopolistically supplied goods tj. Define

$$\omega_t^D = \frac{\phi(p_{t+1}^M - \lambda w_{t+1}) x_{t+1}^M}{(1+r_t) w_t}, \tag{24}$$

which is the present-value expected profit, evaluated by physical labor, from an invention in period t. By (20) and (21), it must hold that

$$\kappa \geq \omega_t^D \tag{25}$$

where equality holds if $Z_t > 0$. This condition implies that ω_t^D must be exceeded by the marginal (labor) cost of innovation, κ, in equilibrium. This shows that ω_t^D may be thought of as a manufacturer's "derived" willingness to pay for an invention in period t, given Z_t.

For this reason, (24) can be transformed into a derived inverse demand function for Z_t. In order to make this transformation, by solving (15), obtain the inverse demand function of the retail sector for a differentiated manufactured good tj, $p_{tj} = p_j(x_{ij}; X_t) = x_{tj}^{-\theta} X_t^{\theta}$. Facing this demand, the monopolistic manufacturer of tj maximizes its profit. Its profit rate can be written as

$$p_{t+1}^M - \lambda w_{t+1} = \frac{\theta}{1-\theta}\lambda w_t. \tag{26}$$

Note that $x_{t+1}^M = (\frac{\lambda w_t}{1-\theta})^{-1/\theta} X_{t+1}$ for a monopolistically manufactured good. By the first order condition for consumers, we have $X_{t+1}/X_t = \beta(1+r_t)$. By using these facts, $p_{t+1}^M - \lambda w_{t+1}$, x_{t+1}^M, and X_{t+1} can be eliminated from (24), which results in

$$\bar{\omega}_t^D = \phi\beta\theta \frac{\left(\frac{\lambda w_{t+1}}{1-\theta}\right)^{1-1/\theta} X_t}{w_t}. \tag{27}$$

By symmetricity, $x_{tj}^C = x_t^C$ and $p_{tj}^C = p_t^C$ hold for all competitively supplied goods tj. Since $p_t^C = \lambda w_t$ by profit maximization, and since $p_j(x_{ij}; X_t) = x_{tj}^{-\theta} X_t^{\theta}$, it holds that $x_t^C = (\lambda w_t)^{-1/\theta} X_t$. By plugging this and $x_t^M = (\frac{\lambda w_t}{1-\theta})^{-1/\theta} X_t$ into the production function of the retail sector, (14), we have

$$w_t = \frac{1}{\lambda}\left(N_{t-1} + \left(1 - \phi\left(1 - \left(\tfrac{1}{1-\theta}\right)^{1-1/\theta}\right)\right) Z_{t-1}\right)^{\frac{\theta}{1-\theta}}. \tag{28}$$

Moreover, by plugging $x_{tj} = x_t^C = (\lambda w_t)^{-1/\theta} X_t$ for any competitively supplied tj and $x_{tj} = x_t^M = (\frac{\lambda w_t}{1-\theta})^{-1/\theta} X_t$ for any monopolistically supplied tj into the labor market clearing condition, (23), we obtain

$$X_t = \frac{\bar{E}\alpha^t - \kappa Z_t}{\lambda^{1-1/\theta}\left(N_{t-1} + (1-\phi(1-\left(\frac{1}{1-\theta}\right)^{-1/\theta}))Z_{t-1}\right) w_t^{-1/\theta}}. \tag{29}$$

Thus, by (28) and (29), (27) can be transformed into the (inverse) derived demand function as follows:

$$\begin{aligned}
&\omega_t^D(Z_t; N_{t-1}, Z_{t-1}) \\
&= \phi\beta\theta\,(1-\theta)^{1/\theta-1} \frac{\xi Z_{t-1} + N_{t-1}}{\zeta Z_{t-1} + N_{t-1}} \frac{\alpha^t \bar{E} - \kappa Z_t}{Z_{t-1} + N_{t-1} + \xi Z_t},
\end{aligned} \tag{30}$$

where

$$\xi = \xi(\theta,\phi) = 1 - \phi\left(1 - (1-\theta)^{1/\theta-1}\right) \tag{31}$$

and

$$\zeta = \zeta(\theta, \phi) = 1 - \phi \left(1 - (1-\theta)^{1/\theta}\right). \tag{32}$$

With (30), (25) can be expressed as a non-autonomous system

$$\omega_t^D(Z_t; N_{t-1}, Z_{t-1}) \leq \kappa, \tag{33}$$

where $Z_t > 0$ only if (33) holds with equality. In summary, our equilibrium stem can be transformed into a two-state-variable dynamical system (2) and (33) together with (3).

The basic working of this system can be illustrated by demand and supply curves in the intellectual property market. See Figure 2. The inverse derived demand curve for inventions in period t is illustrated by curve D_t, which is the graph of (30). The supply curve is the horizontal line through κ, S, which may be thought of the social marginal cost of an invention. Given Z_{t-1} and N_{t-1}, the equilibrium number of inventions, Z_t, is determined at the intersection between demand curve D_t and supply curve S. The cumulative number of differentiated products, N_t, is also determined by (2) as the sum of Z_{t-1} and N_{t-1}.

Note that if the demand curve, D_t, lies below the marginal cost line, κ, no inventions are made in equilibrium, i.e., $Z_t = 0$. This leads to our first main result, which shows that no matter how low the level of intellectual property protection ($\phi > 0$) is, an economy will eventually reach a point at which efforts to make inventions are economical.

Theorem 1 *Let $\kappa > 0$ and $\theta > 0$. For any $\phi > 0$, there is t along the equilibrium path from a given initial condition, (Z_0, N_0), such that $Z_t > 0$ so long as $\alpha > 1$.*

Proof. Suppose $Z_\tau = 0$ for all $\tau = 0, 1, 2, ..., t-1$. Then, by (30), the derived demand curve in period t is

$$\omega_t^D = \frac{\phi\beta\theta\,(1-\theta)^{1/\theta - 1}\,(\alpha^t \bar{E} - \kappa Z_t)}{(N_0 + \xi Z_t)}.$$

Since the vertical intercept of this demand curve must lie above κ (the marginal cost of innovation), it must hold that $Z_t > 0$ if t is sufficiently large. ■

As is shown by curve D_1 in Figure 2, new inventions will not initially take place if the protection of intellectual properties is sufficiently small. If new inventions do not take place for a sufficiently large number of periods, however, the horizontal intercept of demand curve D_t must go above κ, at which point new inventions will take place. This can be interpreted in a broader context than in the modern history of industrial revolution if α is treated as representing the speed of accumulation in human knowledge.

Proposition 3 *No matter how weak the protection of intellectual properties is, innovation will eventually take place so long as human knowledge accumulates exogenously.*

The above result reveals the first driving force of industrial revolution cycles and long waves, Harrod-neutral exogenous technological progress, α. Equation (30) shows that exogenous technological progress shifts the derived willingness to pay for a new invention, ω_t^D, or the derived demand for inventions upwards, as α^t increases.

The second driving force of industrial revolution cycles and long waves is endogenous innovation, which is captured in (30) by Z_{t-1}. An increase in the number of inventions in period $t-1$, Z_{t-1}, increases the number of differentiated products that exist in period t, $N_t = N_{t-1}+Z_{t-1}$, which tends to reduce the derived willingness to pay for an invention in period t, ω_t^D, lowering the third term on the right-hand side of demand function (30), $\frac{\alpha^t \bar{E}-\kappa Z_t}{Z_{t-1}+N_{t-1}+\xi Z_t}$. This is because, as the number of differentiated goods increases, the demand for new products gets reduced, for the demand for each final consumption good gets watered down by an increase in the selection of goods. Note that this effect of an increase in Z_{t-1} is modified by the second term of demand function (30), $\frac{\xi Z_{t-1}+N_{t-1}}{\zeta Z_{t-1}+N_{t-1}}$, which makes the full dynamics of the model highly complicated.

In our model, industrial revolution cycles and long waves are created by the combined effect of Harrod-neutral exogenous technological progress, α^t, and endogenous technological progress through innovation, Z_{t-1}. In the period in which no new innovation is made, the derived demand for inventions increases constantly, which will eventually lead to a period in which new innovation will be made. Once new innovation is made, however, it will bring down the demand for inventions throughout the future periods. Once this effect shifts the derived demand curve for new inventions below the marginal cost curve at κ, a period of no innovation will re-emerge.

5 Kondratieff-like Long Waves

In this section, we will demonstrate the existence of Kondratieff-like irregular long waves in our model. Before doing so, it may be useful diagrammatically to illustrate dynamics in our model. For this purpose, redefine the effective numbers of inventions (cumulative and flow), n_t and z_t, as follows:

$$n_t = \frac{N_t}{\phi\alpha^t} \text{ and } z_t = \frac{Z_t}{\phi\alpha^t}. \tag{34}$$

Moreover, define a new state variable

$$y_t = \frac{1}{\alpha}(n_t + z_t). \tag{35}$$

With this state variable together with n_t, (2), (3), (25), and (30) can be transformed into an autonomous system as follows:

$$\begin{cases} n_{t+1} &= y_t \\ y_{t+1} &= \frac{1}{\alpha}\left[y_t + \max\left\{0, f(n_t, y_t; \theta, \phi)\right\}\right] \end{cases} \tag{36}$$

with initial conditions $\bar{n} = \bar{N}/\phi$ and $\bar{y} = \frac{1}{\alpha}(\bar{N}/\phi + \bar{Z}/\phi)$, where

$$f(n_t, y_t; \theta, \phi) = \frac{\frac{\bar{E}\beta\theta(1-\theta)^{1/\theta-1}}{\kappa}\frac{\xi(\alpha y_t - n_t)+n_t}{\zeta(\alpha y_t - n_t)+n_t} - y_t}{\phi\beta\theta(1-\theta)^{1/\theta-1}\frac{\xi(\alpha y_t - n_t)+n_t}{\zeta(\alpha y_t - n_t)+n_t} + \xi}. \tag{37}$$

System (36) consists two different phases: the innovation phase, in which innovation actually takes place, and the no-innovation phase, in which no innovation takes place. As (35) shows, innovation takes place in period t if and only if $z_t > 0$. This implies that, in Figure 3, line $\Lambda : y_t = \frac{1}{\alpha}n_t$ illustrates the no-innovation phase; if (n_t, y_t) is on line Λ, no innovation takes place in period t. The region above line Λ illustrates the innovation phase. The state variable vector must lie either in the innovation phase (above line Λ) or in the no-innovation phase (on line Λ).

Which of the two phases a state variable vector, (n_{t+1}, y_{t+1}), falls in is determined by the position of the state variable vector in the previous period, (n_t, y_t). That is, if $f(n_t, y_t) \leq 0$, (n_{t+1}, y_{t+1}) lies in the no-innovation phase; in Figure 2, $f(n_t, y_t) \leq 0$ implies that the derived demand curve, D_{t+1}, lies below or on the marginal cost of an invention, κ. If $f(n_t, y_t) > 0$, (n_{t+1}, y_{t+1}) lies in the innovation phase.

In order to illustrate this mechanism, it is necessary to find the region in which $f \leq 0$. Towards this end, by using $\xi > \zeta$, $f(n_t, y_t; \theta, \phi) \leq 0$ if and only if

$$n_t \geq \frac{\alpha\left(\frac{\bar{E}\beta\theta}{\kappa}(1-\theta)^{1/\theta-1}\right)^2 \frac{\xi-\zeta}{1-\zeta}\frac{1-\xi}{1-\zeta}}{(1-\zeta)y_t - \frac{\bar{E}\beta\theta}{\kappa}(1-\theta)^{1/\theta-1}(1-\xi)} - \frac{\alpha\zeta}{1-\zeta}y_t + \alpha\frac{\bar{E}\beta\theta}{\kappa}(1-\theta)^{1/\theta-1}\frac{\xi-\zeta}{(1-\zeta)^2}. \tag{38}$$

The boundary of (38) can be expressed by a function of n_t, say $y_t = b(n_t; \theta, \phi)$. In Figure 3, this function is illustrated by curve $B^{(\theta,\phi)}$. Denote

$$G^{(\theta,\phi)} = \{(n_t, y_t) > 0 : \frac{1}{\alpha} n_t \leq y_t \leq b(n_t; \theta, \phi)\}. \quad (39)$$

The next lemma shows that whether (n_t, y_t) lies above or below line $B^{(\theta,\phi)}$ determines whether (n_{t+1}, y_{t+1}) lies in the innovation phase or in the no-innovation phase.

Lemma 1 *The equilibrium dynamical system,* (2), (3), (25), *and* (30)*, can be transformed as follows:*

$$(n_{t+1}, y_{t+1}) = \begin{cases} (y_t, \frac{1}{\alpha} y_t + f(n_t, y_t; \theta, \phi) & \textit{if } (n_t, y_t) \in G^{(\theta,\phi)} \\ (y_t, \frac{1}{\alpha} y_t) & \textit{if } (n_t, y_t) \notin G^{(\theta,\phi)} \end{cases} \quad (40)$$

Proof. This holds since $(n_t, y_t) \in G^{(\theta,\phi)}$ if and only if $f(n_t, y_t; \theta, \phi) > 0$. ■

In what follows, we may regard (40), rather than (36), as an equilibrium dynamical system. This system has a unique fixed point, which is denoted as $(n^{(\theta,\phi)}, y^{(\theta,\phi)})$; by construction, $n^{(\theta,\phi)} = y^{(\theta,\phi)}$. Moreover, steady state $(n^{(\theta,\phi)}, y^{(\theta,\phi)})$ lies in the interior of region $G^{(\theta,\phi)}$. As the next lemma shows, this steady state can be a saddle point (a proof requires a tedious sequence of calculations, which is omitted here).

Lemma 2 *System* (40) *has a unique steady state, which is a saddle-point if and only if*

$$\beta < \frac{\alpha-\phi(\alpha-1)\left(1-(1-\theta)^{1/\theta}\right)}{\alpha-\phi(\alpha-1)\left(1-(1-\theta)^{1/\theta-1}\right)} \frac{1-(1-\theta)^{1/\theta-1}-\frac{\alpha}{\alpha+1}\frac{1}{\phi}}{\theta(1-\theta)^{1/\theta-1}} - \frac{2\alpha}{(\alpha+1)\left(\alpha-\phi(\alpha-1)\left(1-(1-\theta)^{1/\theta-1}\right)\right)}. \quad (41)$$

An important feature of our model is that, unlike the standard optimal growth model, all the solutions to the equilibrium dynamical system, (40), are equilibrium paths, starting from their respective initial points, even if the equilibrium system is saddle-point stable. This is because system (40) is an equilibrium dynamical system, corresponding to (8), rather than an Euler system, corresponding to (7).

Focus on the case in which (41) is satisfied (so that system (40) is saddle-point stable). In that case, any equilibrium that starts off the stable manifold will fluctuate irregularly and endlessly. This is because, as Figure 3 shows, region $G^{(\theta,\phi)}$ is bounded and because the system has a unique steady state, $(n^{(\theta,\phi)}, y^{(\theta,\phi)})$. As a result, an equilibrium from a point off the stable manifold can never approach to the steady state. Nor does it diverge off to the infinity; once the state variable vector leaves region $G^{(\theta,\phi)}$, it must return to the region along the no-innovation line, Λ.

In order to explain this process, in Figure 3, the stable and unstable manifolds of system (40) are illustrated by M^s and M^u. Suppose that the initial state, $(\bar{n}, \bar{y})$, happens to be on the unstable manifold, M^u. Then, as is shown in Figure 3, state variable (n_t, y_t) moves cyclically on the unstable manifold, M^u. If the unstable manifold, M^u, intersects the boundary $B^{(\theta,\phi)}$, the state variable will eventually go out of region $G^{(\theta,\phi)}$. When that happens, it jumps to point X on line Λ, rather than point X' on the unstable manifold, in period $t+1$. It then moves along line Λ until it returns to region $G^{(\theta,\phi)}$, say in period T; in Figure 3, it is illustrated in such a way that the state variable returns to region $G^{(\theta,\phi)}$ in the next period at X''. After this point, X'', the state variable returns to the interior of $G^{(\theta,\phi)}$. Because a graphical analysis of a two-state variable saddle system is beyond the scope of this study, we do not discuss how the state variable moves after point X''.

This explains that our model generates Kondratieff-like long waves, in which a period of no innovation emerges repeatedly and is followed by that of active innovation. We assume that the length of a single period may be equal to the period in which a patent is protected. In that case, fluctuations in our model may be at least forty year cycles.

Proposition 4 *If condition* (40) *is satisfied, innovation activities on every equilibrium path but the saddle path exhibit slow cycles like those of Kondratieff's long waves.*

Proposition 4 shows a result similar to that of Matsuyama (2001), who demonstrates the existence of a locally unstable steady state in a model in which endogenous innovation is incorporated into the standard optimal growth model. In the subsequent part of this study, in contrast, we demonstrate that industrial-revolution-like phenomena will recur over the time horizon by following a chaotic dynamical system.

6 Existence of Chaotic Industrial Revolution Cycles

In this section, we will show that industrial revolution cycles and Kondratieff-like long waves can coexist along a single equilibrium path. For this purpose, we will focus on the limit case in which the elasticity of substitution is infinitely large (or $\theta \to 0$). This is because, in the limit, the system boils down to a dynamical system with a single state variable, to which standard results on ergodic chaos (Lasota and Yorke (1973) and Sato and Yano (2013)) can be applied. In the mathematical literature, as is well known, a characterization as sharp as those results is not available for the case of two state variables. The existing characterizations for single variable chaotic dynamics make it possible to explain industrial revolution cycles in a more precise manner.

As the out set, note that, as $\theta \to 0$,

$$(1-\theta)^{1/\theta-1} \to \frac{1}{e},\ \xi \to 1-\phi(1-\frac{1}{e}),\ \text{and}\ \zeta \to 1-\phi(1-\frac{1}{e}), \tag{42}$$

where e is the base for natural logarithm. Since $1/\theta$ is the elasticity of substitution between manufactured goods, as (26) shows, the profit rate of a monopolistic manufacturer becomes 0 as $\theta \to 0$. If, therefore, θ alone becomes small, equilibrium condition (33) boils down to a trivial condition $\kappa \geq 0$, with which an equilibrium path is not well defined. However, if the invention cost, κ, becomes smaller at the same time as $\theta \to 0$, the limit system remains to be non-trivial. If, for example, $\kappa = \theta\gamma$, (33) becomes $\omega_t^D(Z_t; N_{t-1}, Z_{t-1})/\theta \leq \gamma$. We will demonstrate that the limit of this system as $\theta \to 0$ is non-trivial and ergodically chaotic.

Let $\kappa = \theta\gamma$. Then, the limit of system (36) becomes

$$\begin{cases} n_{t+1} = y_t \\ y_{t+1} = \eta(y_t), \end{cases} \tag{43}$$

where

$$\eta(y_t) = \begin{cases} -\frac{1}{\alpha}\frac{\phi(e-1)}{e-\phi(e-1)}y_t + \frac{1}{\alpha}\frac{\frac{\bar{E}\beta}{\gamma}}{e-\phi(e-1)} & \text{if } y_t \leq b^o \\ \frac{1}{\alpha}y_t & \text{if } y_t > b^o. \end{cases} \tag{44}$$

and

$$b^o = \frac{\bar{E}\beta}{\gamma e}. \tag{45}$$

In the n-y space, as is shown in Figure 4, this system can be characterized by two lines: Line H^o, which has the slope equal to $-\frac{e-1}{\alpha}\frac{\phi}{\phi+(1-\phi)e}$, and line Λ, which is the ray from the origin with a slope equal to $\frac{1}{\alpha}$. It can be easily checked that lines H^o and Λ intersect each other at $y_t = b^o$, which gives the minimum of η. If (y_t, y_{t+1}) lies above line B^o on line H^o, (y_{t+1}, y_{t+2}) must lie on Λ. If (y_t, y_{t+1}) lies below line B^o on line H^o, (y_{t+1}, y_{t+2}) will also lie on line H^o.

As Figure 4 shows, this implies that all solution paths will be confined in the region between $y_L = \eta(b^o)$ and $y_H = \eta^2(b^o)$. Denote the interval between these two points as I, i.e.,

$$I = \{y : y_L \leq y \leq y_H\}. \tag{46}$$

Note that

$$y_L = \frac{\bar{E}\beta}{\gamma}\frac{1}{\alpha e} \text{ and } y_H = \frac{\bar{E}\beta}{\gamma}\frac{\alpha e - \phi(e-1)}{e - \phi(e-1)}\frac{1}{\alpha^2 e} \quad . \tag{47}$$

In what follows, we restrict the region of function η to I. Then, $\eta : I \to I$ is a dynamical system on a closed interval. The steady state of the limit system, (43), is given by

$$y^* = \frac{\bar{E}\beta}{\gamma}\frac{1}{\alpha e - \phi(e-1)(\alpha - 1)}. \tag{48}$$

As Figure 4 shows, system $\eta : I \to I$ is unimodal but not expansive. The system, η, does not, therefore, satisfy Lasota-Yorke-Kowalski-Li's well-known condition for ergodic chaos. By Sato and Yano's first theorem (Proposition 1), however, a set of parameter values may be determined in such a way that η is strong ergodic chaos.

Theorem 2 *The limit system as $\theta \to 0$, $\eta : I \to I$, is strong ergodic chaos if the exogenous technological progress rate, $\alpha > 1$, satisfies*

$$\alpha < \sqrt{\frac{\phi(e-1)}{e - \phi(e-1)}}. \tag{49}$$

Proof. We will first prove that η is an ergodic chaos under (49). Let $y_t = \eta^t(y_0)$. Note that for any $\alpha > 1$,

$$\eta(y_H) < b^o, \tag{50}$$

which follows from the fact that $\frac{\alpha^2+\alpha}{\alpha^2+\alpha+1}$ is increasing in $\alpha > 1$. By (50), (y_t, y_{t+1}) cannot lie on the upward-sloping segment of the graph of η for two, or more, consecutive periods. Thus, either $(y, \eta(y))$ or $(\eta(y), \eta^2(y))$, or both, must lie on the downward-sloping part of $\eta(y)$. Since the slope of the downward-sloping part is $\frac{1}{\alpha}\frac{\phi(e-1)}{e-\phi(e-1)}$ in absolute value, and since that of the upward-sloping part is $\frac{1}{\alpha} < 1$, by (49),

$$\left|\frac{d\eta^2(y)}{dy}\right| \geq \frac{1}{\alpha^2}\frac{\phi(e-1)}{e-\phi(e-1)} > 1, \tag{51}$$

which implies that η^2 is expansive. Thus, by Proposition 1, η is an ergodic chaos. ■

Theorem 2 implies that industrial revolution cycles and Kondtratieff-like long waves may coexist along a single equilibrium path. This is because, under condition (49), the graph of equilibrium dynamical system η is captured by segment $X_L X^o X_H$ in Figure 1. If $n_0 = b^o$, as is explained in Section 2, an industrial-revolution-like phenomenon emerges in periods 2 and 9; other than those periods, an equilibrium path exhibits shorter, and less drastic, Kondtratieff-like long waves. Around the steady state, an equilibrium path makes almost no fluctuations, as in periods 4 and 5 and the economy grows almost at the steady state rate, α. Sufficiently far away from the steady state, the economy fluctuates slowly, as in periods 6, 7 and 8. After an industrial-revolution-like phenomenon occurs, a deep recession tends to follow, as in periods 3 and 10 (see Yano (2009, 2010) and Furukawa and Yano (2013), who stress the importance of such phenomena in the real world).

In order to examine Theorem 2 more closely, focus on the case in which intellectual properties are fully protected ($\phi = 1$). Given $\phi = 1$, condition (49) becomes

$$\alpha < \sqrt{e-1} (\approx 1.315). \tag{52}$$

Since α is assumed to be any number larger than 1, the above theorem guarantees that our limit equilibrium system, (43), can actually be strong ergodic chaos.

Given (52), Theorem 2 proves the possibility of an industrial revolution in our sense. That is, if an equilibrium path starts from $y_0 \geq b^o$ such that $\eta(y_0) \leq y^*$, then, $(y_0, y_1) = (y_0, \eta(y_0))$ lies on the upward sloping part of the graph of η, and $(y_1, y_2) = (y_1, \eta(y_1))$ will lie on the downward sloping part on $y_L \leq y_1 \leq y^*$. Since this implies that $\rho_1 = 1/\alpha$ and $\rho_2 > 1$, an industrial revolution of scale ρ_2 occurs in the second period. The maximum possible scale of an industrial revolution is given by $\rho_{\max} = \eta(y_L)/y_L$, which implies, by (44),

$$\rho_{\max} = e - \frac{e-1}{\alpha}. \tag{53}$$

Although this shows the possibility of an isolated industrial revolution, it does not guarantee that of persistent industrial revolution cycles. For the sake of explanation, let μ be the invariant measure associated with strong ergodic chaos; an invariant measure is uniquely determined in strong ergodic chaos because the ergodic identity, (10), holds for Lebesgue-almost every initial state (see Sato and Yano (2013)). As (10) shows, equilibrium path y_t will eventually lie in the support of μ; in other words, the support of invariant measure μ is the attractor of the equilibrium path from Lebesgue-almost every initial state. Because, however, the invariant measure is endogenously determined in ergodic chaos, the support of invariant measure cannot, in general, be specified, unless it is explicitly derived. This implies that, in strong ergodic chaos, a solution will wander around over the support of invariant measure, where exactly it will wander around is unknown. This necessitates one more step of proof in order to guarantee that persistent industrial revolution cycles may exist.

Towards this end, define

$$\mathbb{N}^* = \left\{ \tau \in \mathbb{N} : \tau \leq \frac{\log\left(\frac{(e+\alpha-1)\alpha^3}{(\alpha-1)((e\alpha^2-(e-1)^2)}\right)}{\log\left(\frac{e-1}{\alpha}\right)} + \frac{1}{2} \right\}. \tag{54}$$

The next result demonstrates the existence of persisting industrial revolution cycles for a range of parameter α, given by (56).

Theorem 3 *Let $\phi = 1$, $\tau^* = \max_\tau \mathbb{N}^*$, and μ be the ergodically chaotic measure associated with the limit dynamical system, η. From Lebesgue-almost every initial state, there will emerge ergodically chaotic industrial revolution cycles of scale at least as large as*

$$\rho_\alpha = \frac{e\alpha^3 - (e-1)(e(\alpha-1)+1)}{(e(\alpha-1)+1)\alpha} \tag{55}$$

if

$$\frac{e-1}{\sqrt{e}}(\approx 1.035) \leq \alpha \leq \sqrt{e-1}(\approx 1.315). \tag{56}$$

The relative frequency of industrial revolution cycles is $\mu([y_L, y^{\rho_\alpha}]) \geq 1/(1+\tau^*)$, where $y^{\rho_\alpha} = \eta(y_H)$.

Proof. Given (56), it may be proved that $\eta(y_H) < y^*$. This implies that no solution can continue to stay in the interval between $\eta(y_H)$ and $\eta^2(y_H)$. Take the case in which the initial state is outside of this interval, $[\eta(y_H), \eta^2(y_H)]$. Then, the solution, y_t, will never fall in the interval over time. Moreover, if (y_{t-1}, y_t) is on the interval between $X^o X^H$, then $y_t \leq \eta(y_H)$ and $y_{t+1} \geq \eta^2(y_H)$. Thus,

$$\rho_{t+1} = n_{t+2}/n_{t+1} = y_{t+1}/y_t \geq \eta^2(y_H)/\eta(y_H).$$

Since $\eta^2(y_H)/\eta(y_H) = \rho_\alpha$, ρ_α is a least scale of industrial revolutions.

Since no equilibrium path can stay on segment $X_L X^o$ forever, without loss of generality, there is y_t such that $y_t > b^o$. Denote $y_{t+\tau} = \eta^\tau(y_t)$. Then, $y_{t+1} = \eta(y_H) < y^*$. By condition (56), it holds that

$$y^* - \eta(y_H) = \frac{\bar{E}\beta}{\kappa} \frac{(\alpha-1)(e\alpha^2 - (e-1)^2)}{(e+\alpha-1)\alpha^3 e} > 0.$$

Thus, by (44), it holds that

$$y_{t+\tau+1} - y^* = -\frac{e-1}{\alpha}(y_{t+\tau} - y^*) \tag{57}$$

for $\tau = 1, 2, ...$ so long as $y_{t+2\tau} \leq b^o$. Thus, we have

$$b^o \geq y_{t+2\tau} - y^* \geq \left(-\frac{e-1}{\alpha}\right)^{2(\tau-1)} (y_{t+2} - y^*) \tag{58}$$

so long as $y_{t+2\tau} \leq b^o$. Since, by (56) and (57), it holds that

$$y_{t+2} - y^* \geq \eta^2(y_H) - y^* = -\frac{e-1}{\alpha}(\eta(y_H) - y^*),$$

(58) implies

$$\left(\frac{e-1}{\alpha}\right)^{2\tau-1} \frac{(\alpha-1)((e\alpha^2 - (e-1)^2)}{(e+\alpha-1)\alpha^3} \leq 1 \tag{59}$$

so long as $y_{t+2\tau} \leq b^o$. The largest $\tau \in \mathbb{N}$ that satisfies this relationship is given by τ^*. Since $\tau = 1 \in \mathbb{N}^*$, $\tau^* \geq 1$, which gives the maximum number of periods in which $(y_\tau, y_{\tau+1})$ can stay on $X_H X^o$. Since, as is shown in the proof of Theorem 2, $(y_\tau, y_{\tau+1})$ can stay on $X_H X^o$ only once, this implies that if $y_0 \notin [\eta(y_H), \eta^2(y_H)]$,

$$\frac{1}{T}\sum_{t=0}^{T-1} \chi_{[b^o, y_H]}(\eta^t(y_0)) \geq \frac{1}{\tau^*+1}. \tag{60}$$

If $(y_\tau, y_{\tau+1})$ stays on $X^o X_H$, $y_{\tau+1} \in [y_L, \eta(y_H)]$. Thus, by (60),

$$\frac{1}{T}\sum_{t=1}^{T} \chi_{[y_L, \eta(y_H)]}(\eta^t(y_0)) \geq \frac{1}{\tau^*+1}.$$

Thus, by Proposition 2,

$$\mu([y_L, \eta(y_H)]) = \lim \frac{1}{T} \sum_{t=1}^{T} \chi_{[y_L, \eta(y_H)]}(\eta^t(y_0)) \geq \frac{1}{\tau^* + 1}$$

for Lebesgue-almost every y_0. This establishes the theorem. ■

Since $\mu([y_L, y^{\rho_\alpha}]) \geq 1/(1+\tau^*) > 0$, the above theorem implies the generic existence of ergodically chaotic industrial revolution cycles, i.e., that ergodically chaotic industrial revolution cycles may occur along the equilibrium path from Lebesgue-almost every initial stock. This result can be explained by the interaction between exogenous and endogenous factors.

The mechanism through which ergodically chaotic equilibrium paths arise in our model is similar to, but much more complicated than, the chaotic equilibrium dynamics revealed by Nishimura and Yano (1995a). In their study, chaotic dynamics is attributed to the existence of a natural upper boundary for capital accumulation and the fact that the consumption good sector is capital intensive while the capital good sector is labor intensive. In our model as well, consumption is capital intensive; we assume that innovation requires only labor while consumption requires both innovation and labor. Moreover, the natural lower boundary exists due to the assumption of irreversibility for knowledge accumulation ($Z_t \geq 0$). Our model, involving two state variables, is more complicated than that of Nishimura and Yano (1995a), which is concerned with a standard one-sector optimal model.

Our result on the limit case shows that the smaller θ, the more likely industrial revolution cycles emerge. This is because, as the next remark shows, $1/\theta$ may be 0 in our limit model can be interpreted as capturing the substitutability between consumption goods and inventions. The larger this substitutability (i.e., the smaller θ), the more likely an equilibrium fluctuates over time between the production of consumption goods and the creation of new inventions, thereby magnifying the scale of industrial revolution.

Remark 1 *As is noted above, θ controls the substitutability on the production side between consumption goods and inventions. This fact can be explained in a simple manner in the case in which all sectors were perfectly competitive and in which a steady state activity is taken on the production side. (Although a similar argument can be made for the general case, we choose not to do so, for that would significantly complicate the expressions below.) If all sectors were perfectly competitive, all differentiated products manufacturers are symmetric. Thus, we may assume that they behave identically, $x_t(j) = x_t$ for all j (or, in other words, $x_t^M = x_t^C = x_t$). By using this* *fact, we may rewrite the production function of the consumption good sector as*

$$X_t = \left[\int_0^{N_t} x_t(j)^{1-\theta} dj\right]^{\frac{1}{1-\theta}} = N_t^{\frac{1}{1-\theta}} x_t. \tag{61}$$

In the case of $\phi = 1$, (23) *implies* $\lambda N_t x_t + \kappa Z_t = \bar{E}\alpha^t$. *Thus,* (61) *implies*

$$x_t = \frac{\theta^{\frac{1}{1-\theta}}}{\lambda} n_t^{\frac{\theta}{1-\theta}} (\frac{\bar{E}}{\theta} - \kappa z_t), \tag{62}$$

where $x_t = X_t/(\alpha^{\frac{1}{1-\theta}})^t$. *Since* $n_t = n$ *and* $z_t = z$ *hold for all* t *in a steady state, it holds that* $n = \frac{1}{\alpha-1} z$. *Thus,* (62) *implies*

$$x = \frac{\theta^{\frac{1}{1-\theta}}}{\lambda(\alpha-1)^{\frac{\theta}{1-\theta}}} z^{\frac{\theta}{1-\theta}} (\frac{\bar{E}}{\theta} - \kappa z), \tag{63}$$

which may be shown to be concave for $\theta < 1/2$. *Thus, given* $\theta < 1/2$, *the steady-state production possibility frontier between* x *and* z *are concave and bowed out in the region in which* z *is sufficiently large. Moreover,* (63) *shows that in the case in which* $\theta = 0$, *in any steady state, consumption* x *and invention* z *are perfectly substitutable on the production side and that they are imperfectly substitutable in the case in which* θ *is close to* 0. *This explains that the smaller* θ, *the more substitutable consumption goods and new inventions are in the production process.*

The above result may be summarized as follows:

Proposition 5 *In the limit case in which consumption and innovation are perfect substitutes, ergodically chaotic industrial revolution cycles may occur under a sufficient protection of intellectual property rights.*

The analysis of the limit case of $\theta \to 0$ is not sufficient to guarantee the existence of industrial revolution cycles in our model. This is because the original equilibrium system with state variables N_t and Z_t is not well defined in the limit in which $\theta = 0$. Despite this, as the next theorem shows, a standard continuity argument can guarantee that the equilibrium system (36) with $\theta > 0$ behaves in much the same fashion as the limit system with $\theta = 0$, (43). (A proof of this result is routine and given in the Appendix.)

Theorem 4 *Let $\phi = 1$. Suppose that (57) is satisfied. Then, there is $\theta' > 0$ such that if $0 < \theta < \theta'$, from Lebesgue-almost every initial point, industrial revolution cycles will emerge.*

This implies the following:

Proposition 6 *Under a sufficient protection of intellectual property rights, industrial revolution cycles may emerge if the substitutability between consumption and innovation is sufficiently large.*

As is shown in Section 3, the exogenous growth factor constantly raises the productivity of labor during the period in which no innovation takes place. This will in turn raise the potential profitability of innovation. As this continues for sufficiently many periods, it reaches a point at which innovation will explode. This provide monopolistic firms with large profit opportunities, which may be thought of as representing the early stage of an industrial revolution. As inventions continues, new firms enter the market, thereby reducing their monopolistic profit opportunities. This will then reduce the incentive for new inventions, thereby slowing down innovation again. Once innovation ceases to take place, an exogenous rise in the productivity of labor will make innovation again profitable and trigger the next industrial revolution. In this process, market quality, measured by the degree of competition, falls during the period in which the number of new inventions bursts and then rises as more new inventions are created.

7 Intellectual Property Protection as a Necessary Condition

In this section, we will demonstrate that intellectual property protection is a necessary condition for the emergence of persistent industrial revolution cycles. In order to prove this result, we will first examine the limit case in which the intellectual property protection rate is $\phi = 0$. Since, as $\phi \to 0$, $\xi \to 1$ and $\zeta \to 1$ by (31) and (32), in the limit, system (36) becomes

$$\begin{cases} n_{t+1} &= y_t \\ y_{t+1} &= \frac{1}{\alpha}\left[y_t + \max\left\{0, \frac{\bar{E}\beta\theta(1-\theta)^{1/\theta-1}}{\kappa} - y_t\right\}\right]. \end{cases} \tag{64}$$

Region $G^{(\theta,\phi)}$ becomes $G^{(\theta,0)}$, which is the region on or below the horizontal line at $y = \frac{\bar{E}\beta\theta(1-\theta)^{1/\theta-1}}{\kappa}$ (or $B^{(\theta,0)}$).

In this limit system, so long as (n_t, y_t) lies outside of region $G^{(\theta,0)}$, (n_{t+1}, y_{t+1}) lies on line Λ (i.e., $y_{t+1} = \frac{1}{\alpha} n_{t+1}$) at $n_{t+1} = y_t$. This implies that the dynamics of (n_{t+1}, y_{t+1}) outside of region $G^{(\theta,0)}$ is captured by line Λ and the 45 degree line in the way illustrated in Figure 5. This implies that state variable (n_t, y_t) will eventually enter region $G^{(\theta,0)}$. Once (n_t, y_t) falls in region $G^{(\theta,0)}$, it holds that $n_{t+1} = y_t$ and $y_{t+1} = \frac{\bar{E}\beta\theta(1-\theta)^{1/\theta-1}}{\alpha\kappa}$. Moreover, $n_{t+2} = \frac{\bar{E}\beta\theta(1-\theta)^{1/\theta-1}}{\alpha\kappa}$ and $y_{t+2} = \frac{\bar{E}\beta\theta(1-\theta)^{1/\theta-1}}{\alpha\kappa}$, which implies that the dynamics of (n_t, y_t) follows the horizontal line at $y^o = \frac{\bar{E}\beta\theta(1-\theta)^{1/\theta-1}}{\alpha\kappa}$.

This demonstrates that in the limit system as $\phi \to 0$, equilibrium dynamics can be illustrated by a one dimensional dynamical system

$$y_{t+1} = \begin{cases} \frac{1}{\alpha} y_t & \text{if } y_t > \frac{\bar{E}\beta\theta(1-\theta)^{1/\theta-1}}{\alpha\kappa} \\ \frac{\bar{E}\beta\theta(1-\theta)^{1/\theta-1}}{\alpha\kappa} & \text{if } y_t \leq \frac{\bar{E}\beta\theta(1-\theta)^{1/\theta-1}}{\alpha\kappa} \end{cases} . \tag{65}$$

As this shows, any solution to the limit system, (64), will eventually reach the steady state, (n^o, y^o), satisfying

$$n^o = y^o = \frac{\bar{E}\beta\theta(1-\theta)^{1/\theta-1}}{\alpha\kappa}. \tag{66}$$

Although an equilibrium monotonically reaches this limit steady state, it is not sufficient for establishing the necessity of intellectual property protection for the emergence of industrial revolution cycles, because the original equilibrium system, consisting of conditions (2), (3), (25), and (30), has only a trivial equilibrium in the case in which intellectual properties are not protected at all ($\phi = 0$).

For this reason, we will prove the necessity of intellectual property protection by focusing on the case in which the intellectual property protection rate, $\phi > 0$, is sufficiently small. That is, we will prove that if $\phi > 0$ is sufficiently small, in a sufficiently far future, new inventions will take place every period along an equilibrium path in system (36). This implies that an industrial revolution in our sense cannot take place in a sufficiently future.

In doing so, one further difficulty arises because the limit system, (64), is not hyperbolic (i.e., one of its characteristic root is 0). As a result, the stability of the steady state in the limit system, in general, may not be preserved with respect to a perturbation in ϕ. Instead, we will prove the Liapunov stability of the interior of region $G^{(\theta,\phi)}$, which implies $z_t > 0$ for any sufficiently large t, i.e., that new inventions will be made in a sufficiently far future.

Towards this end, we will first prove the next lemma, which implies that once inventions are made in a particular period along an equilibrium path, they will be made in every subsequent period, so long as ϕ is sufficiently small.

Lemma 3 *There is* $\phi' > 0$ *such that if* $0 < \phi < \phi'$ *and* $f(n_t, y_t, \theta, \phi) > 0$, *then* $f(n_{t+1}, y_{t+1}, \theta, \phi) > 0$.

Proof. In order to highlight the fact that ξ and ζ depend on ϕ, we denote $\xi = \xi(\phi)$ and $\zeta = \zeta(\phi)$. Since $f(n_{t-1}, y_{t-1}; \theta, \phi) > 0$, by (36), $n_{t+1} = y_t$ and

$$y_{t+1} = \frac{1}{\alpha}\left[y_t + \frac{\frac{\bar{E}\beta\theta(1-\theta)^{1/\theta-1}}{\kappa}\frac{\xi(\phi)(\alpha y_t - n_t)+n_t}{\zeta(\phi)(\alpha y_t - n_t)+n_t} - y_t}{\phi\beta\theta(1-\theta)^{1/\theta-1}\frac{\xi(\phi)(\alpha y_t - n_t)+n_t}{\zeta(\phi)(\alpha y_t - n_t)+n_t} + \xi(\phi)}\right].$$

Denote

$$\Delta_{t+1} = \frac{\bar{E}\beta\theta(1-\theta)^{1/\theta-1}}{\kappa}\frac{\xi(\phi)(\alpha y_{t+1} - n_{t+1})+n_{t+1}}{\zeta(\phi)(\alpha y_{t+1} - n_{t+1})+n_{t+1}} - y_{t+1} \ .$$

Then, by (36), the sign of $f(n_{t+1}, y_{t+1}; \theta, \phi)$ is equal to that of Δ_{t+1}. Since

$$\begin{aligned}\Delta_{t+1} \quad = \quad & \frac{\bar{E}\beta(1-\theta)^{1/\theta-1}}{\kappa}\frac{\xi(\phi)(\alpha y_{t+1} - y_t)+y_t}{\zeta(\phi)(\alpha y_{t+1} - y_t)+y_t} - \frac{1}{\alpha}\frac{\frac{\bar{E}\beta\theta(1-\theta)^{1/\theta-1}}{\kappa}\frac{\xi(\phi)(\alpha y_t - n_t)+n_t}{\zeta(\phi)(\alpha y_t - n_t)+n_t}}{\phi\beta\theta(1-\theta)^{1/\theta-1}\frac{\xi(\phi)(\alpha y_t - n_t)+n_t}{\zeta(\phi)(\alpha y_t - n_t)+n_t}+\xi(\phi)} \\ & -\frac{1}{\alpha}\left[1 - \frac{1}{\phi\beta\theta(1-\theta)^{1/\theta-1}\frac{\xi(\phi)(\alpha y_t - n_t)+n_t}{\zeta(\phi)(\alpha y_t - n_t)+n_t}+\xi(\phi)}\right]y_t \ ,\end{aligned}$$

and since $\xi(\phi) \to 1$ and $\zeta(\phi) \to 1$ as $\phi \to 0$,

$$\Delta_{t+1} \to \left(1 - \frac{1}{\alpha}\right)\frac{\bar{E}\beta\theta(1-\theta)^{1/\theta-1}}{\kappa} > 0.$$

Thus, the lemma holds. ■

The main result of this section, establishing the necessity of intellectual property protection, follows from this lemma together with the continuity of system (36) in (n_t, y_t, ϕ).

Theorem 5 *Let* (n_t, y_t) *be a solution from* $(\bar{n}, \bar{y})$ *to the equilibrium system* (36) *with* $\phi > 0$. *There are* $\phi' > 0$ *and* t' *such that if* $0 < \phi < \phi'$, $z_t = \alpha y_t - n_t > 0$ *for all* $t > t'$.

Proof. In order to prove this theorem, define

$$g_2(n, y; \theta, \phi) = \frac{1}{\alpha}\left[y + \max\{0, f(n, y; \theta, \phi)\}\right] \tag{67}$$

and $g(n, y; \theta, \phi) = (y, g_2(n, y; \theta, \phi))$. Then, by construction, $(n_t, y_t) = g^t(\bar{n}, \bar{y}; \theta, \phi)$ is the equilibrium path from $(\bar{n}, \bar{y})$. Moreover, $z_t = \alpha y_t - n_t$.

We will first demonstrate that $g(n, y; \theta, \phi)$ is continuous in (n, y, ϕ) at $(n^o, y^o, 0)$. For this purpose, it suffices to demonstrate that for any $\varepsilon > 0$, there is $\delta > 0$ such that $|(n, y, \phi) - (n^o, y^o, 0)| < \delta$ implies

$$|g_2(n, y; \phi, \theta) - g_2(n^o, y^o; 0, \theta)| < \varepsilon. \tag{68}$$

Note that

$$f(n, y; \theta, \phi) \to \frac{\bar{E}\beta\theta(1-\theta)^{1/\theta-1}}{\kappa} - y = f(n, y; \theta, 0). \tag{69}$$

Take the case of $f(n^o, y^o; \theta, 0) > 0$, i.e., $\frac{\bar{E}\beta\theta(1-\theta)^{1/\theta-1}}{\kappa} > y^o$. Then, since f is continuous in (n, y, ϕ), there is $\delta > 0$ such that $|(n, y, \phi) - (n^o, y^o, 0)| < \delta$ implies $f(n, y; \theta, \phi) > 0$. This implies that

$$g_2(n, y; \theta, \phi) = \frac{1}{\alpha}\left[y + f(n, y; \theta, \phi)\right]$$

and

$$g_2(n^o, y^o; \theta, 0) = \frac{1}{\alpha}\left[y^o + f(n^o, y^o; \theta, 0)\right]$$

Thus, (68) holds for a properly chosen $\delta > 0$. Next, take the case of $\frac{\bar{E}\beta\theta(1-\theta)^{1/\theta-1}}{\kappa} < y^o$. Then, there is $\delta > 0$ such that $|(n, y, \phi) - (n^o, y^o, 0)| < \delta$ implies $f(n, y; \phi, \theta) < 0$. In this case,

$$g_2(n, y; \theta, \phi) = g_2(n^o, y^o; \theta, 0) = 0, \tag{70}$$

in which case (68) holds. Finally, take the case of $\frac{\bar{E}\beta\theta(1-\theta)^{1/\theta-1}}{\kappa} = y^o$. If $f(n, y; \phi, \theta) \leq 0$, then (70) holds again, in which case (68) holds. If $f(n, y; \theta, 0) > 0$, by (69), for any $\varepsilon > 0$, there is $\delta > 0$ such that $|(n, y, \phi) - (n^o, y^o, 0)| < \delta$ implies

$$0 < f(n, y; \theta, \phi) < \varepsilon.$$

In this case, by (67),

$$g_2(n, y; \theta, \phi) = \frac{1}{\alpha}\left[y + f(n, y; \theta, \phi)\right]$$

and

$$g_2(n^o, y^o; \theta, 0) = \frac{1}{\alpha} y.$$

Thus,

$$|g_2(n, y; \theta, \phi) - g_2(n^o, y^o; \theta, 0)| = f(n, y; \theta, \phi) < \varepsilon.$$

Thus, (68) holds.

In order to complete the proof, define $(n_t^o, y_t^o) = g^t(\bar{n}, \bar{y}; \theta, 0)$ and $z_t^o = \alpha y_t^o - n_t^o$. Then, by (65), there are t' and $\varepsilon' > 0$ such that $z_{t'}^o = (\alpha - 1)y^o > \varepsilon'$. Since, by the continuity of g_2, $g^{t'}(\bar{n}, \bar{y}; \theta, \phi)$ is continuous at $\phi = 0$, for there is $\phi' > 0$ such that $0 < \phi < \phi'$ implies

$$\left| g^{t'}(\bar{n}, \bar{y}; \theta, \phi) - (n_{t'}^o, y_{t'}^o) \right| < \varepsilon'/2,$$

which implies $g_2^{t'}(\bar{n}, \bar{y}; \theta, \phi) > \varepsilon'/2$. By Lemma 3, this implies $z_t > 0$ for all $t > t'$. ■

The main result of this section may be summarized as follows:

Proposition 7 (necessity of intellectual property protection) *Without an institutional protection of intellectual properties, industrial revolution cycles do not occur.*

Appendix: Proof of Theorem 4

In this appendix, we prove Theorem 4. In order to simplify our presentation, express (36) as

$$(n_{t+1}, y_{t+1}) = F(n_t, y_t; \theta) = (F_1(n_t, y_t; \theta), F_2(n_t, y_t; \theta)); \tag{71}$$

i.e.,

$$F_1(n_t, y_t; \theta) = y_t \tag{72}$$

and

$$F_2(n_t, y_t; \theta) = \frac{1}{\alpha}\left(y_t + \max\{0, f(n_t, \alpha y_t - n_t; \theta)\}\right). \tag{73}$$

The next lemma implies that boundary $B^{(\theta,\phi)}$ of region $G^{(\theta,\phi)}$ in Figure 3 uniformly converges to line B^o in Figure 4.

Lemma 4 *For any* $\varepsilon > 0$, *there is* $\theta' > 0$ *such that if* $0 < \theta < \theta'$, $\left|b^\theta(n) - b^o\right| < \varepsilon$ *for any* $n \geq 0$.

Proof. As is illustrated in Figure 3, curve $B^{(\theta,\phi)}$ is decreasing in n and asymptotic to a horizontal line. Moreover, this horizontal line converges to line $B^o(: y = b^o)$ in Figure 4, as $\theta \to 0$. Thus, in order to prove the lemma, it suffices to show that the vertical intercept of curve $B^{(\theta,\phi)}$ converges to b^o, as $\theta \to 0$.

Towards this end, note that curve $B^{(\theta,\phi)}$ is given by the equation resulting from (38) holding with equality. Therefore, by setting $n_t = 0$ in this equality, we obtain the vertical intercept of curve $B^{(\theta,\phi)}$,

$$y^\theta = \frac{\bar{E}\beta}{\gamma}\frac{(1/\theta - 1)^{1/\theta - 1}}{(1/\theta)^{1/\theta - 1}}\frac{\xi - \zeta + (1-\xi)\zeta}{(1-\zeta)\,\zeta}.$$

By (42), it may be proved that $y^\theta \to b^o$ as $\theta \to 0$. ■

The next lemma demonstrates that the graph of the general system stays in a neighborhood of that of the limit system. In Figure 4, this graph lies in a neighborhood of segment $X_L X^o$ and on segment line Λ in a neighborhood of segment $X^o X_H$. Toward this end, take an arbitrarily small $\varepsilon > 0$, and define interval $K = [y_L - \varepsilon, y_H + \varepsilon]$ in $\mathbb{R}_+$. Then, $intK \supset I = [y_L, y_H]$. Restrict F to K^2. We will prove the next lemma:

Lemma 5 *For any* $\varepsilon > 0$, *there are* $\theta' > 0$ *and* $\delta' > 0$ *such that if* $0 < \theta < \theta'$ *and* $|(\nu, \mu) - (n, y)| < \delta'$, $|F(\nu, \mu; \theta) - F(n, y; 0)| < \varepsilon$ *for any* $(n, y) \in K^2$.

Proof. By definition, $F_1(\nu,\mu;\theta) - F_1(n,y;0) = \mu - y$. Thus,

$$|F(\nu,\mu;\theta) - F(n,y;0)| \le (1+\frac{1}{\alpha})\,|\mu - y| \\ + \frac{1}{\alpha}\left|\max\{0, f(\nu,\alpha\mu-\nu;\theta)\} - \max\{0, f(n,\alpha y - n;\theta)\}\right|.$$

Denote

$$\Delta = \left| \frac{\frac{\bar{E}\beta(1-\theta)^{1/\theta-1}}{\gamma}\frac{\alpha\xi\mu+(1-\xi)\nu}{\alpha\zeta\mu+(1-\zeta)\nu} - \mu}{\phi\beta\theta(1-\theta)^{1/\theta-1}\frac{\alpha\xi\mu+(1-\xi)\nu}{\alpha\zeta\mu+(1-\zeta)\nu} + \xi} - \frac{\frac{\bar{E}\beta}{\gamma e} - y}{1-\phi(1-\frac{1}{e})} \right|$$

Then, by the definition of F, (71), the following holds:

If $\frac{\bar{E}\beta}{\gamma}\frac{(1/\theta-1)^{1/\theta-1}}{(1/\theta)^{1/\theta-1}}\frac{\alpha\xi\mu+(1-\xi)\nu}{\alpha\zeta\mu+(1-\zeta)\nu} - \mu \ge 0$ and $\frac{\bar{E}\beta}{\kappa e} - y \ge 0$,

$$|\max\{0, f(\nu,\alpha\mu-\nu;\theta)\} - \max\{0, f(n,\alpha y-n;\theta)\}| = \Delta.$$

If $\frac{\bar{E}\beta}{\gamma}\frac{(1/\theta-1)^{1/\theta-1}}{(1/\theta)^{1/\theta-1}}\frac{\alpha\xi\mu+(1-\xi)\nu}{\alpha\zeta\mu+(1-\zeta)\nu} - \mu \ge 0$ and $\frac{\bar{E}\beta}{\gamma e} - y < 0$,

$$|\max\{0, f(\nu,\alpha\mu-\nu;\theta)\} - \max\{0, f(n,\alpha y-n;\theta)\}| \\ = \left| \frac{\frac{\bar{E}\beta}{\gamma}\frac{(1/\theta-1)^{1/\theta-1}}{(1/\theta)^{1/\theta-1}}\frac{\alpha\xi\mu+(1-\xi)\nu}{\alpha\zeta\mu+(1-\zeta)\nu} - \upsilon}{\phi\beta\theta\frac{(1/\theta-1)^{1/\theta-1}}{(1/\theta)^{1/\theta-1}}\frac{\alpha\xi\mu+(1-\xi)\nu}{\alpha\zeta\mu+(1-\zeta)\nu} + \xi} \right| \le \Delta.$$

If $\frac{\bar{E}\beta}{\gamma}\frac{(1/\theta-1)^{1/\theta-1}}{(1/\theta)^{1/\theta-1}}\frac{\alpha\xi\mu+(1-\xi)\nu}{\alpha\zeta\mu+(1-\zeta)\nu} - \mu < 0$ and $\frac{\bar{\bar{E}}\beta}{\gamma e} - y \ge 0$

$$|\max\{0, f(\nu,\alpha\mu-\nu;\theta)\} - \max\{0, f(n,\alpha y-n;\theta)\}| \\ = \frac{1}{1-\phi(1-\frac{1}{e})}\left|\frac{\bar{\bar{E}}\beta}{\gamma e} - y\right| \le \Delta.$$

If $\frac{\bar{\bar{E}}\beta}{\gamma}\frac{(1/\theta-1)^{1/\theta-1}}{(1/\theta)^{1/\theta-1}}\frac{\alpha\xi\mu+(1-\xi)\nu}{\alpha\zeta\mu+(1-\zeta)\nu} - \mu < 0$ and $\frac{\bar{\bar{E}}\beta}{\gamma e} - y < 0$,

$$|\max\{0, f(\nu,\alpha\mu-\nu;\theta)\} - \max\{0, f(n,\alpha y-n;\theta)\}| = 0 \le \Delta.$$

Since $\Delta \to 0$ and $|\mu - y| \to 0$ as $|(\nu,\mu) - (n,y)| \to 0$, the lemma is proved. ∎

Recall that $(\bar{n},\bar{y})$ is the initial condition of the system. Lemmas 4 and 5 imply that industrial revolution cycles emerge in the original model, so long as θ is sufficiently small.

Proof of Theorem 4. Let $(n_t^\theta, y_t^\theta) = F^t(n_0, y_0, \theta)$ and $(n_t^0, y_t^0) = F^t(n_0, y_0, 0)$. Then, (n_t^θ, y_t^θ) and (n_t^0, y_t^0) are the equilibrium path in the original system with θ and the solution to the limit system at $\theta = 0$. Let y^ρ be such that

$\eta(y^\rho)/y^\rho = \rho$. Then, for any $\delta > 0$, there is ρ, $\rho_{\max} - \delta < \rho \leq \rho_{\max}$ such that $\mu([y_L^0, y^\rho]) > 0$. Thus, from Lebesgue-almost every initial point, it is possible to choose ρ in such a way that $y_t^0 \in [y_L^0, y^\rho]$ implies $y_{t+1}^0 > b^o$. Since F^t is continuous in θ, by Lemma 4, it is possible to choose θ' so that $0 < \theta < \theta'$ implies $y_{t+1}^\theta > b^\theta(n_{t+1}^\theta)$. This implies $(n_{t+2}^\theta, y_{t+2}^\theta)$ is on line Λ.

Now, denote $(n_{t+2+\tau}^{\theta 0}, y_{t+2+\tau}^{\theta 0}) = F^\tau(n_{t+2}^\theta, y_{t+2}^\theta, 0)$ and $(n_{t+2+\tau}^\theta, y_{t+2+\tau}^\theta) = F^\tau(n_{t+2}^\theta, y_{t+2}^\theta, \theta)$. As is shown in the proof of Theorem 3, $(n_{t+2+\tau}^{\theta 0}, y_{t+2+\tau}^{\theta 0})$ must go above line B^o within $\tau^* + 1$ periods. Thus, by the continuity of F^τ in θ, $(n_{t+2+\tau}^\theta, y_{t+2+\tau}^\theta)$ must go above curve B^θ within $\tau^* + 1$. Once $(n_{t+2+\tau}^\theta, y_{t+2+\tau}^\theta)$ goes above curve B^θ at τ', $(n_{t+2+\tau}^\theta, y_{t+2+\tau}^\theta)$ must lie on line Λ. Since this process repeats endlessly, the theorem is proved. ■

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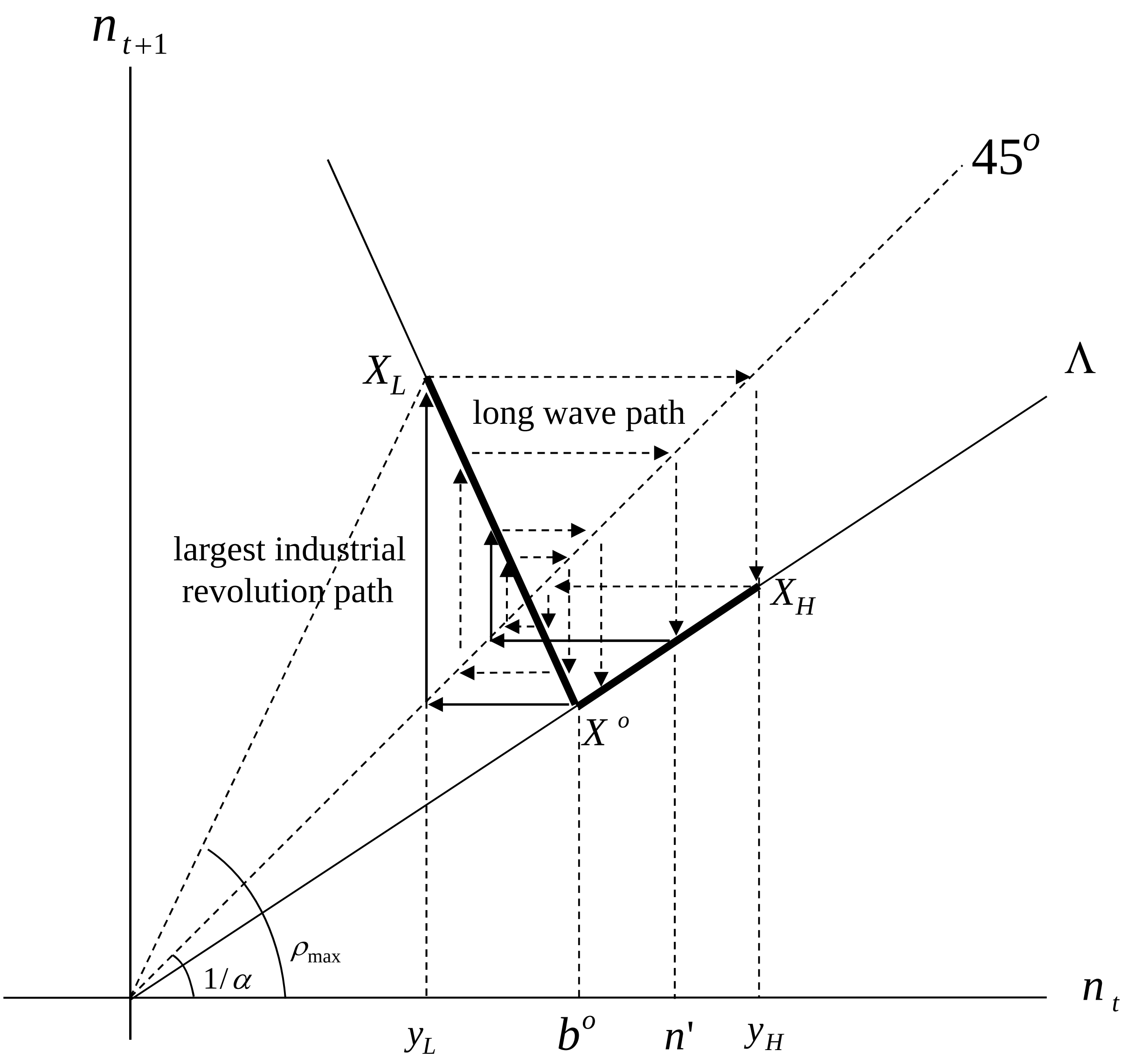
n_{t+1}
45^o
Λ
X_L
long wave path
largest industrial
revolution path
X_H
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y_L
b^o
n'
y_H

Figure 1

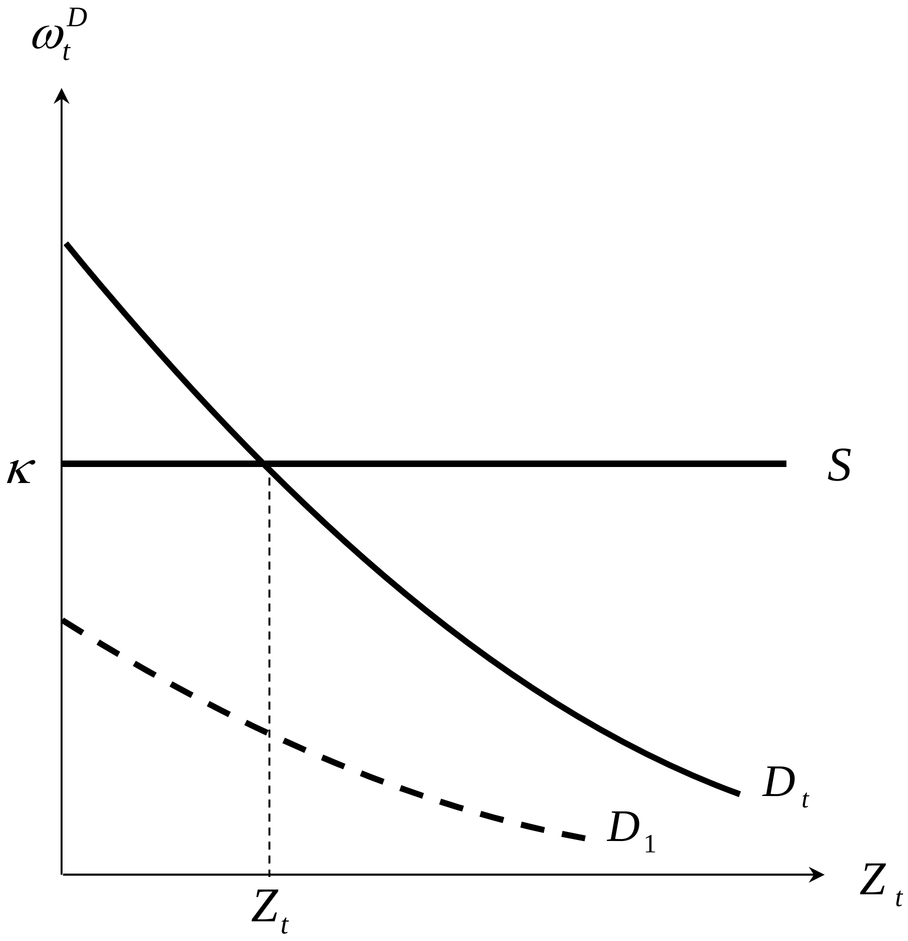
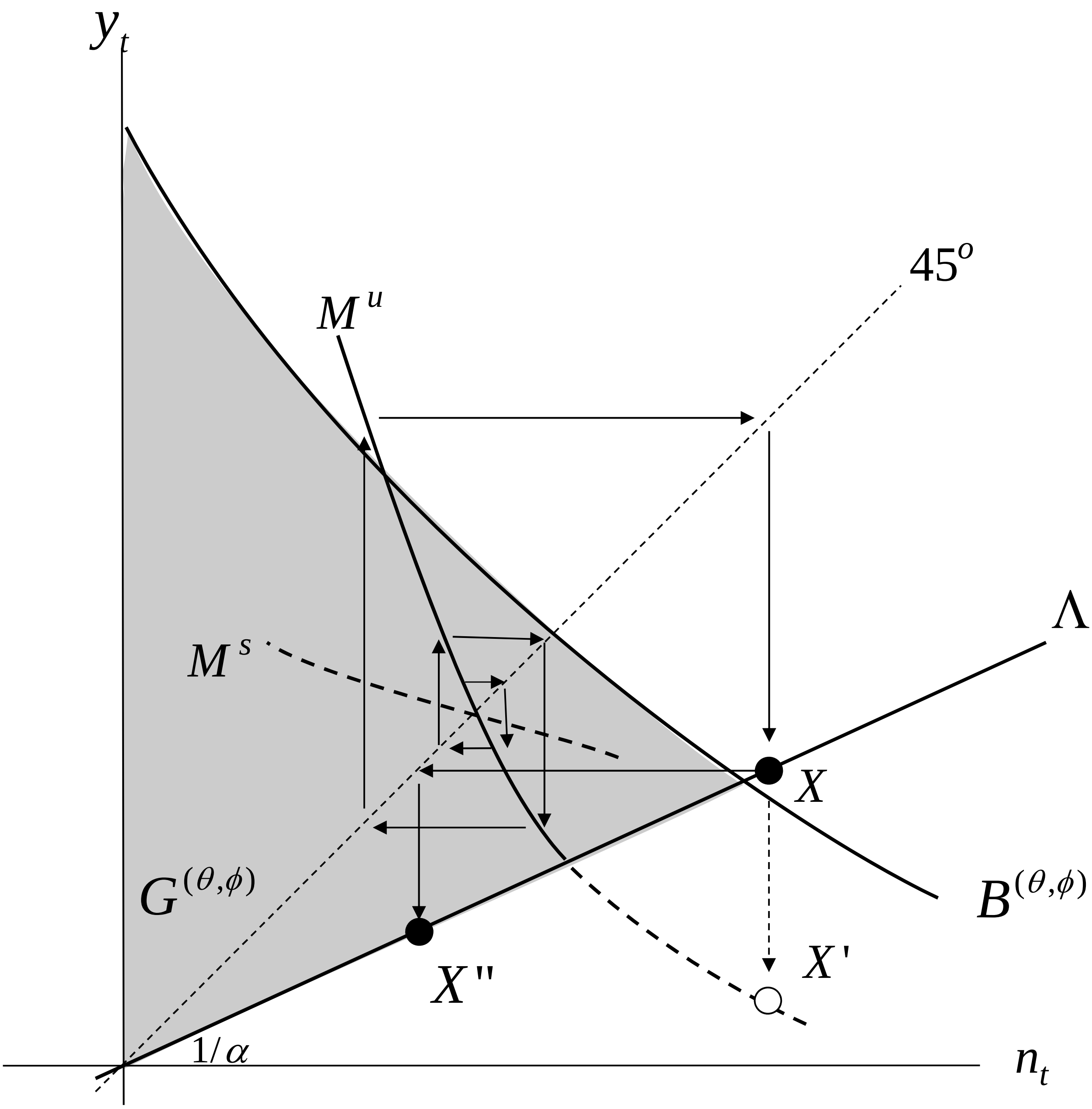


Figure 2



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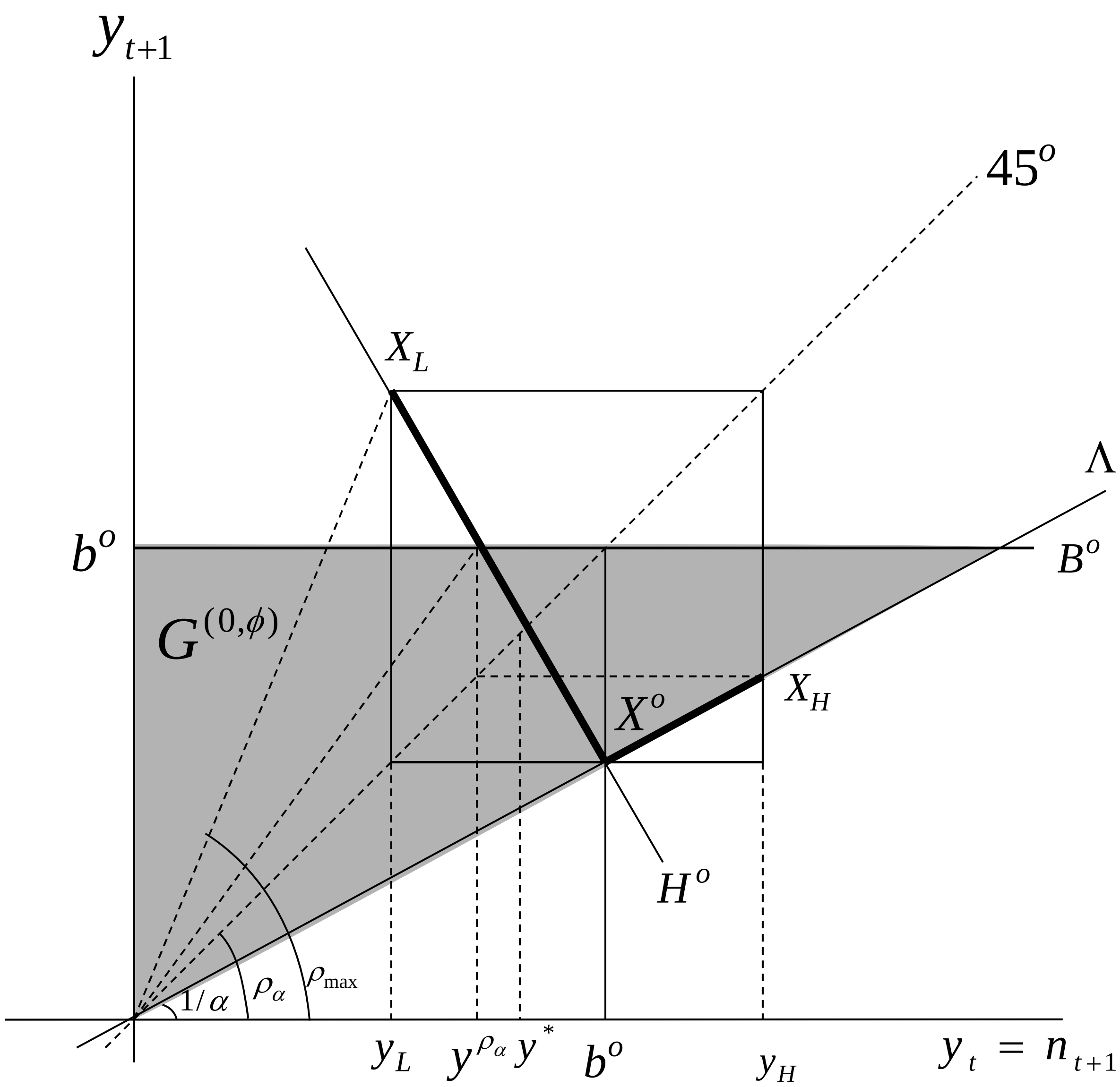
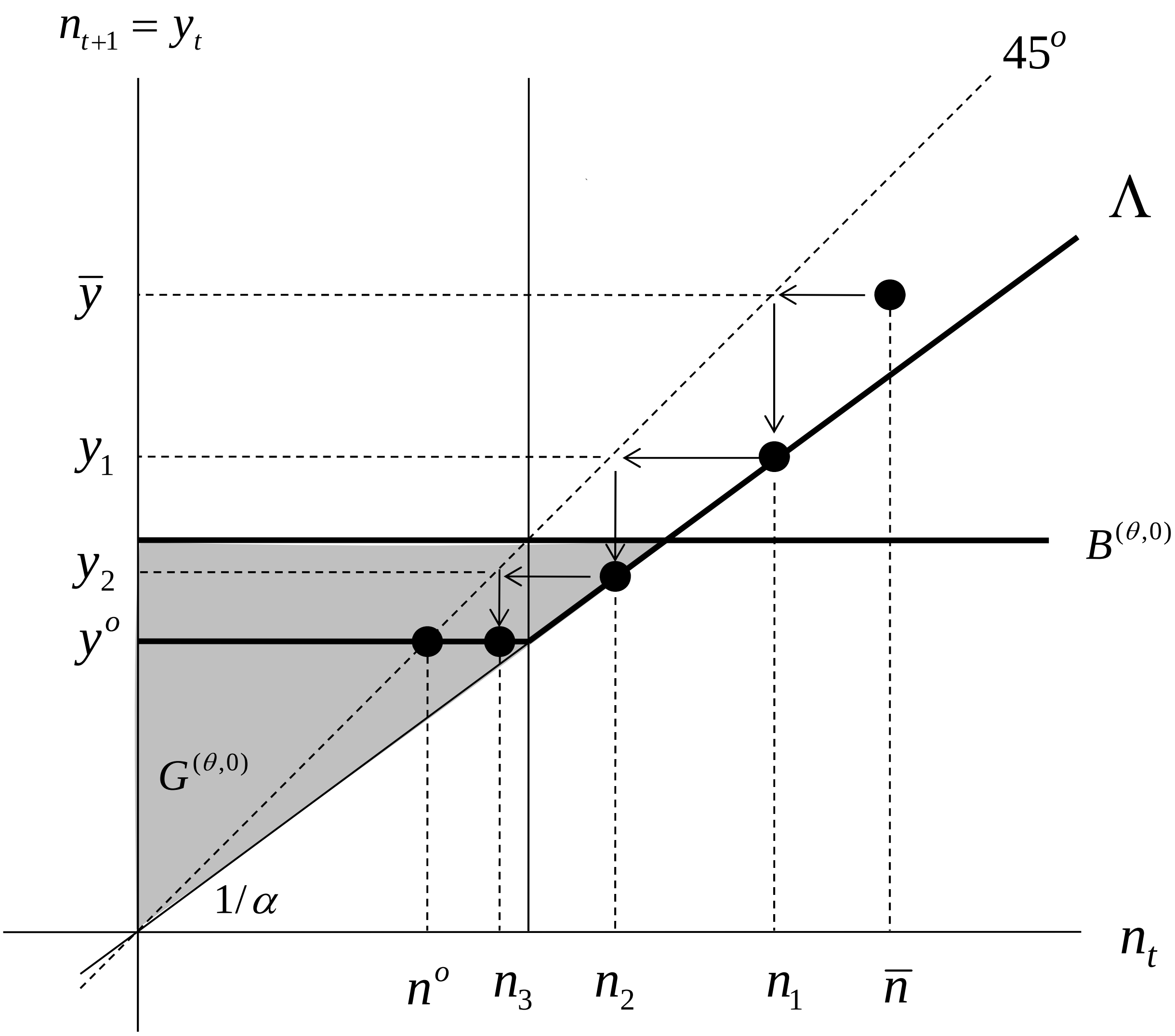


y_{t+1}
45^o
X_L
Λ
b^o
B^o
$G^{(0,\phi)}$
X_H
X^o
H^o
ρ_{max}
ρ_α
$1/\alpha$
y_L
y^{ρ_α}
y^*
b^o
y_H
$y_t = n_{t+1}$

Figure 4


$n_{t+1} = y_t$
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y_1
$B^{(\theta,0)}$
y_2
y^o
$G^{(\theta,0)}$
$1/\alpha$
n_t
n^o
n_3
n_2
n_1
$\bar{n}$

Figure 5